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On the Absence of Order in 2-Dimensional Systems with Compact Symmetry

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Abstract An alternative proof for the generalization to any compact Lie group of the absence of an ordered phase in one and two dimensional classical systems is provided using the original Bogoliubov inequality.

The absence of spontaneous symmetry breaking for quantum spin systems with low dimensionality is a well known result[1]. Similar conclusions were reached for superfluid or superconducting ordering[2]. The validity of this property for classical spin systems was proved by Mermin[3] in a theorem according to which it is impossible to have an ordered phase in one and two dimensions. This property has also been verified for the continuum limit[4] showing therefore to be of great relevance in both statistical and field theory[5].

The extension of this theorem to any continuous symmetry was done using for the U(1) group either an argument dealing with the energy of configurations[6] or a more mathematical approach[7], followed by a generalization based on the properties of compact Lie groups. A more direct proof has been given[8] based on a generalization of Bogoliubov inequality. The purpose of this note is to reach similar conclusions using the original Bogoliubov inequality, following the steps of Mermin's proof, applied to the group of interest. This procedure has the advantage of being more straightforward and intuitive from a physical standpoint. The analysis made by Mermin was applied to systems invariant under a U(1)

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or SU(2) global symmetry based on a Poisson brackets Lie algebra. We first generalize the theorem for models which assume, for each site of a D-dimensional lattice, a set of n classical variables satisfying a compact semi-simple Lie algebra of Poisson brackets. (The extension to non semi-simple groups will be treated at the end). For this purpose it is essential to show that Poisson brackets may be calculated directly in terms of these *spin* quantities without relying on canonical variables. We are then able to prove that, in the absence of external fields, the statistical average of any quantity in the set which defines the rank of the algebra, vanishes.

Let us take at each of the N lattice sites the set of *spin* variables F_a, $a = 1...n$ obeying the semi-simple Lie algebra

$$\{F_a, F_b\} = -i\, C^d_{ab}\, F_d \tag{1}$$

where the Poisson bracket {,} would correspond to $-i$ times the commutator [,] in a quantum version of the model. To give a meaning to any Poisson bracket it is sufficient to take for granted the existence of an expression of F_a in terms of m pairs of canonical variables q_k, p_k with no need of its explicit form. In fact for any two functions of F_a, we find

$$\{A,B\} = \sum_{k=1}^{m} \left(\frac{\partial A}{\partial q_k}\frac{\partial B}{\partial p_k} - \frac{\partial A}{\partial p_k}\frac{\partial B}{\partial q_k} \right) = \sum_{a,b} \frac{\partial A}{\partial F_a}\frac{\partial B}{\partial F_b}\{F_a, F_b\} \tag{2}$$

where the chain rule for derivatives has been used. It is possible to prove that the last expression for $\{A,B\}$ in eq. *(2)* satisfies all the requirements to be a Poisson bracket, including the Jacobi identity.

We remark that the Bogoliubov inequality continues to hold without surface terms. To show this it is straightforward to reproduce the calculation of ref. 3 to give

$$\langle A^*\{B,H\}\rangle = \frac{i}{2\beta}\sum_{\underset{\sim}{R}'} \int \prod_{a,\underset{\sim}{R}} d\,F_a(\underset{\sim}{R})\;\cdot\; F^2(\underset{\sim}{R}')\,\frac{\partial A^*}{\partial F_d(\underset{\sim}{R}')}\,C^d_{bc}\,\frac{\partial}{\partial F_b(\underset{\sim}{R}')}\left(P\,e^{-\beta(H-F)}\right)\frac{\partial B}{\partial F_c(\underset{\sim}{R}')} \tag{3}$$

where H is the hamiltonian, F the free energy, $F^2 = F_a F_a$ the quadratic Casimir quantity and < > means a statistical average with a weight P. For simplicity we will take

$$P = \prod_{\underset{\sim}{R}} \delta(F - F(\underset{\sim}{R})) \tag{4}$$

thus fixing the Casimir quantity at each site to a constant value, but more general conditions could be envisaged, as e.g. in Mermin's original paper[3].

The total antisymmetry of the structure constants of any semi--simple Lie group(+) allows the reordering of the factors in the integral of eq. (3) to give

$$<A^*\{B,H\}> = -\frac{1}{\beta}<\{A^*,B\}> \tag{5}$$

explicitely proving the absence of the surface terms which otherwise would appear on the right hand side of this equation. Now using Schwartz inequality for the scalar product $(A,C) = <A^* C>$, it follows

$$<|A|^2> \geq \frac{1}{\beta} \frac{|<\{B,A^*\}>|^2}{<\{B,\{B^*,H\}\}>} \tag{6}$$

which is the inequality due to Bogoliubov.

Going back to our Poisson bracket algebra of dimension n and rank R we rewrite it in terms of quantities

$$H_i, i = 1\ldots\ell \quad \text{and} \quad E_\alpha, \alpha = \pm 1, \ldots \pm \frac{n-\ell}{2}$$

which mimic the mutually commuting hermitian and step operators of the quantum version[10]

$$\begin{aligned} \{H_i,H_j\} &= 0 \\ \{H_i,E_\alpha\} &= -i\, r_i(\alpha)\, E_\alpha \\ \{E_\alpha,E_{-\alpha}\} &= -i\, r^i(\alpha)\, H_i \\ \{E_\alpha,E_\beta\} &= -i\, N_{\alpha,\beta}\, E_{\alpha+\beta} \end{aligned} \tag{7}$$

(+) The total antisymmetry of the structure constants for a compact semi-simple Lie group follows from the recognition of $(X,Y) = -\mathrm{Tr}(ad\, X, ad\, Y)$ as a scalar product[9]. Then $([X,Y],Z) = (X,[Y,Z])$ and taking an orthonormal basis F_a, $a=1,\ldots n$, we have $C^d_{ab} = (F_d, \sum_\ell C^\ell_{ab} F_\ell) = C^b_{da} = -C^b_{ad}$.

where the real roots $r_i(\alpha)$ and coefficients $N_{\alpha,\beta}$ satisfy

$$
\begin{aligned}
r_i(\alpha) &= r^i(\alpha) \\
r_i(-\alpha) &= - r_i(\alpha) \\
\sum_\alpha r_i(\alpha) r_j(\alpha) &= (\ell + 1)\delta_{ij} \\
N_{\alpha,\beta} = -N_{-\alpha,-\beta} &= N_{\beta,-\alpha-\beta} = N_{-\alpha-\beta,\alpha}
\end{aligned}
\tag{8}
$$

We notice that our quantities obey $H_i^* = H_i$ and $E_\alpha^* = E_{-\alpha}$, which are the classical counterparts of $H_i^+ = H_i$ and $E_\alpha^+ = E_{-\alpha}$ for the quantum version. In terms of these variables the Casimir quantity becomes

$$
F^2 = \sum_\alpha E_\alpha E_{-\alpha} + \sum_i H_i^2 \tag{9}
$$

We assume a hamiltonlan invariant under global transformations of the Lle group, describlng the coupling of the generators ateach site with those of the neighbouring ones constrained to constant F^2 in accordance with eq. (4), with the addition of external fields g_i interactlng with the real generators H_i, i.e.

$$
\begin{aligned}
H = &- \sum_{\underset{\sim}{R},\underset{\sim}{R}'} J(\underset{\sim}{R}-\underset{\sim}{R}')\Big[\sum_\alpha E_\alpha(\underset{\sim}{R})\, E_\alpha^*(\underset{\sim}{R}') + \vec{H}(\underset{\sim}{R}) \cdot \vec{H}(\underset{\sim}{R}')\Big] \\
&- \sum_{\underset{\sim}{R}} \vec{g} \cdot \vec{H}(\underset{\sim}{R})
\end{aligned}
\tag{10}
$$

We have introduced a vector notatlon for $\vec{H} = \{H_i,\ i=1,\ldots\ell\}$.

In order to apply the Bogoliubov inequality, eq. (6), we choose

$$
A = B = E_{-\alpha}(-\underset{\sim}{k}) = \sum_{\underset{\sim}{R}} E_{-\alpha}(\underset{\sim}{R})\, e^{i\,\underset{\sim}{k}\cdot\underset{\sim}{R}} \tag{11}
$$

One then readily obtains

$$
\{B,A^*\} = i \sum_{\underset{\sim}{R}} \vec{r}(\alpha)\cdot\vec{H}(R) \tag{12}
$$

$$
\begin{aligned}
\{B,\{B^*,H\}\} = 2 &\sum_{\underset{\sim}{R},\underset{\sim}{R}'} J(\underset{\sim}{R}-\underset{\sim}{R}')\big[1 - e^{i\,\underset{\sim}{k}\cdot(\underset{\sim}{R}-\underset{\sim}{R}')}\big] \\
&\cdot\Big[\sum_{\alpha'} N^2_{\alpha,\alpha'}\ E_{\alpha'}(\underset{\sim}{R})\ E_{-\alpha'}(\underset{\sim}{R}') + \vec{r}(\alpha)\cdot\vec{H}(\underset{\sim}{R})\vec{r}(\alpha)\cdot\vec{H}(\underset{\sim}{R}') \\
&+ \vec{r}^{\,2}(\alpha)E_\alpha(\underset{\sim}{R}')E_{-\alpha}(\underset{\sim}{R})\Big] + \vec{g}\cdot\vec{r}(\alpha) \sum_{\underset{\sim}{R}} \vec{H}(R)\cdot\vec{r}(\alpha)
\end{aligned}
\tag{13}
$$

Expressing eq. (13) in terms of the Fourier transforms we get

$$\{B,\{B^*,H\}\} = \frac{2}{N}\sum_{\underset{\sim}{k}'} \left[J(\underset{\sim}{k}') - J(\underset{\sim}{k}'-\underset{\sim}{k})\right]$$

$$\cdot \left[\sum_{\alpha'} N^2_{\alpha,\alpha'} |E_{\alpha'}(\underset{\sim}{k}')|^2 + \vec{r}^2(\alpha)|E_\alpha(\underset{\sim}{k}')|^2 + (\vec{r}(\alpha).\vec{H}(\underset{\sim}{k}'))^2\right]$$

$$+ \vec{g} . \vec{r}(\alpha)\ \vec{r}(\alpha) . \sum_{\underset{\sim}{R}} \vec{H}(\underset{\sim}{R}) \qquad (14)$$

Since from eq. (5)

$$<\{B,\{B^*,H\}\}> = \beta <|\{B, H\}|^2>$$

is positive definite, using the bounds

$$\frac{1}{N^2}\sum_{\underset{\sim}{k}} |E_\alpha(\underset{\sim}{k})|^2 < F^2$$

$$\frac{1}{N^2}\sum_{\underset{\sim}{k}} [\sum_{\alpha'} N^2_{\alpha,\alpha'} |E_{\alpha'}(\underset{\sim}{k})|^2 + \vec{r}^2(\alpha)\ E_\alpha(\underset{\sim}{k})|^2$$

$$+ (\vec{r}(\alpha) . \vec{H}(\underset{\sim}{k}))^2] < \nu F^2 \qquad (15)$$

with ν some positive number, and the approximation

$$J(\underset{\sim}{k}') - J(\underset{\sim}{k}'-\underset{\sim}{k}) \simeq (\frac{\underset{\sim}{k}^2}{2} - \underset{\sim}{k}'.\underset{\sim}{k})\ \frac{1}{D}\sum_{\underset{\sim}{R}} R^2\ J(\underset{\sim}{R}) \qquad (16)$$

valid for $J(\underset{\sim}{R})$ very peaked(≠) around $\underset{\sim}{R} = 0$, one arrives at

$$F^2 \geqslant \frac{1}{\beta}(\vec{h}.\vec{r}(\alpha))^2\ \frac{1}{N}\sum_{\underset{\sim}{k}} \left[k^2\ \nu\ F^2 \sum_{R} R^2\ J(\underset{\sim}{R}) + \vec{g}.\vec{r}(\alpha)\vec{h}.\vec{r}(\alpha)\right]^{-1} \qquad (17)$$

where $\vec{h}=<\vec{H}(R)>$ defines the order parameter. Note that the negative terms on the right hand side of eq. (16) do not contribute to the statistical average of eq. (14) since

$$\sum_{\underset{\sim}{k}} \underset{\sim}{k} <|E_\alpha(\underset{\sim}{k})|^2> = \sum_{\underset{\sim}{k}} \underset{\sim}{k}<H^2_i(\underset{\sim}{k})> = 0 \qquad (18)$$

(≠) Clearly an exponential behaviour for $J(\underset{\sim}{R})$ is sufficient for the approximation in eq. (16) to be valid. The proof of ref.6 shows that a power law $R^{-\alpha}$ with a ⩾ 4 is enough.

Now the same conclusion obtained by Mermin can be read out of eq. (17) in the limit of vanishing external field, i.e. $\vec{g}=0$. The right hand side of this equation diverges for $D=1,2$ unless the order parameter $\vec{h}$ vanishes. This comes out from the consideration of the R independent root vectors $\vec{r}(\alpha)$. E.g. for SU(3) the order parameter has two components which can be chosen to be the hypercharge and the third component of isospin.

We have thus proved that Mermin theorem is valid not only for the traditional U(1) or SU(2) spin systems but for whatever set of classical variables obeying the algebra of any compact semi-simple Lie group.

The extension to a compact non semi-simple Lie group rests on the theorem[**] which proves that any algebra of a compact Lie group may be written as the direct sum of its centre and the orthogonal complement which is an invariant semi-simple sub algebra. This theorem is based on the existence of a scalar product invariant in the sense $([X,Y],Z) = (X,[Y,Z])$ which generalizes the trace definitions of footnote[(+)].

Since the Mermin's theorem is valid for an abelian algebra and we have proved it for any semi simple Lie algebra, it is obviously valid for the direct sum of the center and its orthogonal complement, i.e. for any compact connected Lie group.

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Resumo

Uma demonstração diferente para a generalização a qualquer grupo de Lie compacto da ausência de fase ordenada em sistemas clássicos a uma e duas dimensões é obtida usando a desigualdade de Bogoliubov original.