

## Influence of Cusps and Intersections on the Wilson Loop in $v$ -Dimensional Space

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Recebido em 26 de Dezembro de 1984

**Abstract** A discussion is given about the influence of cusps and intersections on the calculation of the Wilson loop in  $v$ -dimensional space. In particular, for the two-dimensional case, it is shown that there are no divergences.

### 1. INTRODUCCION

If we want to describe the dynamics of any system, a good choice of a set of coordinates is of primary importance. In gauge theory the choice of good coordinates is equally important. They not only make the treatment simpler but also provide insight into the dynamics. In non-Abelian gauge theories a good choice of coordinator is the expectation value of a non-local gauge invariant operator, the well known Wilson Loop (WL) defined by an appropriate Euclidean functional integral.

We can write the Wilson loop as a perturbative series if we want to make non-trivial calculations involving it. In this case, however, there appear contour integrals which are divergent. For smooth (i.e. differentiable) and simple (i.e. non-intersecting) loops, it was shown<sup>2</sup> that those integrals are renormalizable in four dimensions, and therefore also the WL, in all orders of perturbation theory. It is, however, insufficient to consider only smooth and simple loops, because for simple loops the equations of motion reduce essentially to their Abelian limit and so contain no information about the crucial non-Abelian dynamics<sup>3</sup>. Thus, it is essential to consider self-intersecting loops and loops with cusps. The renormalization program for such loops in four dimensions was carried out by Brandt, Neri and Sato<sup>4</sup>.

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Work partially supported by CNPq (Brazilian Government Agency).

In this paper we investigate the influence of angles and other geometrical factors (lengths, for example) in the residue of the singularity that appears in the contour integrals, by computing them as a function of the number of dimensions  $\nu$  of the space. For this we make use of the dimensional regularization method for treating the infinities that appear in our computations. A particular discussion is given for the two dimensional case.

For simplicity we limit the discussion to planar curves in  $\nu$ -dimensional spaces and consider only the second order term of the perturbative series of the WL, which is given by

$$W^{(2)}(C) = -\frac{1}{2} \oint_C dx^\alpha \oint_C dy^\beta D_{\alpha\beta}(x-y) \quad (1)$$

where  $D_{\alpha\beta}(x-y)$  is the free gluon propagator, which in  $\nu$  dimensions is given by

$$D_{\alpha\beta}(x-y) = \int \frac{d^4 k}{(2\pi)^\nu} \left\{ \frac{g_{\alpha\beta}}{k^2} - A \frac{k_\alpha k_\beta}{k^4} \right\} e^{ik \cdot x} \quad (2)$$

Here  $A$  is a gauge fixing constant.

In configuration space eq. (2) leads to

$$D_{\alpha\beta}(x-y) = \frac{\Gamma\left(\frac{\nu-2}{2}\right)}{4\pi^{\nu/2}} \left\{ g_{\alpha\beta} |x-y|^{2-\nu} - \frac{A}{2(\nu-4)} \partial x_\alpha \partial y_\beta |x-y|^{4-\nu} \right\} \quad (3)$$

Since in our calculations we consider open curves, we use the notation  $V(C)$  instead of  $W^{(2)}(C)$ . Then, putting eq. (3) into eq. (1), with  $W^{(2)}(C)$  exchanged by  $V(C)$ , we obtain

$$V(C) = -\frac{\Gamma\left(\frac{\nu-2}{2}\right)}{8\pi^{\nu/2}} \left\{ \iint_C dx^\alpha dy^\alpha |x-y|^{2-\nu} + \frac{A}{\nu-4} |a-b|^{4-\nu} \right\} \quad (4)$$

$a, b$  being the end points of  $C$ .

Sometimes it is convenient to compute  $V(C)$  in momentum space. To do this we proceed in analogy with Abud, Bollini and Giambiagi<sup>7</sup>, defining a linear functional over the curve  $C$ ,

$$f_C^\alpha(k) = \int_C e^{ik \cdot x} dx^\alpha \quad (5)$$

With use of eq. (5) we can write  $V(C)$  as

$$V(C) = - \frac{1}{2(2\pi)^{\nu}} \int d^{\nu}k \left[ \frac{g_{\alpha\beta}}{k^2} - \frac{2k_{\alpha}k_{\beta}}{(3-\nu)k^4} \right] f_C^{\alpha}(k) f_C^{\beta*}(k) \quad (6)$$

where we have taken  $A = 2/3-\nu$  (dimensional gauge) as in ref. 7.

Let us now suppose that the curve  $C$  is contained in some subspace with  $n$  dimensions. As in ref. 7, we can decompose the vector  $k$  as  $k_{\mu} = k_{\mu}^{\perp} + \hat{k}_{\mu}$ , where  $\hat{k}_{\mu}$  is the projection of  $k_{\mu}$  over the subspace containing  $C$ , and  $k^{\perp}$  is the orthogonal component of  $k$ . Now, performing the integration over  $k^{\perp}$  in eq. (6) and using the fact that  $f_C(k)$  only depends on  $\hat{k}$ , we get<sup>7</sup>

$$V(C) = - \frac{\Gamma\left(1 - \frac{\nu-n}{2}\right)}{2^{\nu+1} \pi^{(\nu+n)/2}} \int d^n \hat{k} (\hat{k}^2)^{\frac{\nu-n}{2} - 1} \left[ g_{\alpha\beta} - \frac{2-\nu+n}{3-\nu} \frac{\hat{k}_{\alpha}\hat{k}_{\beta}}{\hat{k}^2} \right] f_C^{\alpha}(\hat{k}) f_C^{\beta*}(\hat{k}) \quad (7)$$

We only consider curves with  $n=2$ , so we can drop the hat over  $k$ , and call the components of  $k$  by  $k_1$  and  $k_2$ . In terms of the integration over  $k_1$  and  $k_2$ ,  $V(C)$  is given by<sup>7</sup>

$$V(C) = \frac{-2\Gamma\left(\frac{4-\nu}{2}\right)}{(4\pi)^{(\nu+2)/2}} \int_{-\infty}^{+\infty} dk_1 \int_{-\infty}^{+\infty} dk_2 (k_1^2 + k_2^2)^{(\nu-4)/2} |f|^2 + \frac{\Gamma(\nu-4) |a-b|^{4-\nu}}{(4\pi)^{(\nu-1)/2} \Gamma[(\nu-i)/2]} \quad (8)$$

## 2. SINGULAR POINTS

We are interested in the effect of cusps and intersections on the calculation of the Wilson loop. Initially, we consider the interaction between two segments with lengths  $L_1$  and  $L_2$  (see fig. 1), which touch one another forming an angle. In accordance with the general formalism, the interaction between the segments, in the dimensional gauge, is

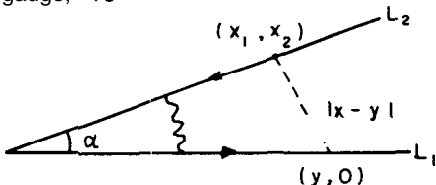


Fig.1 - One segment just touching another at the end points.

$$\begin{aligned}
 V = & \frac{-2\Gamma[(\nu-2)/2]}{8\pi^{\nu/2}} \int_{\text{segl seg2}} dx^\beta \int dy^\beta \left[ (y-x_1)^2 + x_2^2 \right]^{(2-\nu)/2} \\
 & - \frac{\Gamma[(\nu-2)/2] (L_1^{4-\nu} + L_2^{4-\nu} - |\vec{L}_1 - \vec{L}_2|^{4-\nu})}{4\pi^{\nu/2} (3-\nu)(4-\nu)}
 \end{aligned} \tag{9}$$

As in ref. 7, we can perform the change of variables

$$\begin{aligned}
 x_1 &= \zeta \cos \alpha \\
 x_2 &= \zeta \sin \alpha \\
 x_3 &= (y - x_1) = \lambda x_3
 \end{aligned} \tag{10}$$

Then, eq. (9) becomes

$$\begin{aligned}
 V(C) = & \frac{-2\Gamma[(\nu-2)/2]}{8\pi^{\nu/2}} \int_{L_2}^0 d\zeta \int_{-1}^{\infty \text{sgn}(\cos \alpha)} d\lambda \zeta^{3-\nu} \cos^2 \alpha (\sin \alpha)^{2-\nu} (1 + \lambda^2 \cot^2 \alpha)^{(2-\nu)/2} \\
 & - \frac{\Gamma[(\nu-2)/2] (L_1^{4-\nu} + L_2^{4-\nu} - |\vec{L}_1 - \vec{L}_2|^{4-\nu})}{4\pi^{\nu/2} (3-\nu)(4-\nu)}
 \end{aligned} \tag{11}$$

Integrating eq. (11) over  $\zeta$ , we obtain

$$\begin{aligned}
 V(C) = & \frac{2\Gamma[(\nu-2)/2]}{8\pi^{\nu/2}} \frac{L_2^{4-\nu}}{(4-\nu)} \cos^2 \alpha (\sin \alpha)^{2-\nu} \int_{-1}^{\infty \text{sgn}(\cos \alpha)} \lambda (1 + \lambda^2 \cot^2 \alpha)^{(\nu-2)/2} \\
 & - \frac{\Gamma[(\nu-2)/2] (L_1^{4-\nu} + L_2^{4-\nu} - |\vec{L}_1 - \vec{L}_2|^{4-\nu})}{4\pi^{\nu/2} (3-\nu)(4-\nu)}
 \end{aligned} \tag{12}$$

From eq.(12), we see that there is a pole at  $\nu=3$  associated with the gauge choice which can be removed; at  $\nu=4$ , there is a pole whose residue is given by<sup>7</sup>

$$R = \frac{(\pi-\alpha)\cot\alpha + 1}{4a^2} \quad (13)$$

For  $\nu=5,6,\dots$  there is no explicit pole. Note that the result (15) is valid when one of the arrows enters and the other leaves the vertex. In the other cases the residue takes a minus sign.

The case  $\nu=2$  will be considered later.

Now, we are interested in the case of intersection, and we take two segments of equal lengths  $L$  (see fig. 2)

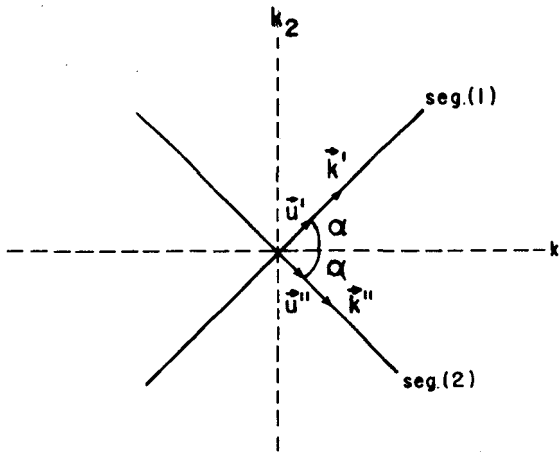


Fig.2 - Two segments that cross each other forming an angle  $2\alpha$ .

From eq. (5) we have the following result for the functional over a segment along the  $k_1$  axis

$$f_{\text{seg}} = \int_{-L/2}^{L/2} e^{ik_1 x_1} dx_1 = \frac{2}{k_1} \sin(k_1 L) \quad (14)$$

or, with the use of the vectorial notation

$$\vec{f}_{\text{seg}(1)} = \frac{2}{k'} \sin(k' \frac{L}{2}) \vec{u}' \quad (15)$$

$$\vec{f}_{\text{seg}(2)} = \frac{2}{k''} \sin(k'' \frac{L}{2}) \vec{u}''$$

where  $u'$  and  $u''$  are unit vectors in the directions of  $\vec{k}'$  and  $\vec{k}''$ , respectively.

It is easy to see that  $k'$ ,  $k''$ ,  $k_1$  and  $k_2$  are related by

$$k'k'' = k_1^2 \cos^2\alpha - k_2^2 \sin^2\alpha \quad (16)$$

From the general formalism, the interaction between the segments, in the Fourier space, is given by

$$V = \frac{-16\Gamma[(4-\nu)/2]}{(4\pi)^{(\nu+2)/2}} \int_{-\infty}^{+\infty} dk_1 \int_{-\infty}^{+\infty} dk_2 \cos(2\alpha) \quad (17)$$

$$\frac{\sin(k_1 \frac{L}{2} \cos \alpha + k_2 \frac{L}{2} \sin \alpha) \sin(k_1 \frac{L}{2} \cos \alpha - k_2 \frac{L}{2} \sin \alpha)}{(k_1^2 \cos^2 \alpha - k_2^2 \sin^2 \alpha) (k_1^2 + k_2^2) (4-\nu)/2}$$

where we have used that

$$|f|^2 = 2f_{\text{seg}(1)} f_{\text{seg}(2)}$$

to take into account the total contribution of the interaction.

In eq. (17) we take the gauge term equal to zero. Using

$$\begin{aligned} & \sin(k_1 \frac{L}{2} \cos \alpha + k_2 \frac{L}{2} \sin \alpha) \sin(k_1 \frac{L}{2} \cos \alpha - k_2 \frac{L}{2} \sin \alpha) = \\ & = \sin^2(k_1 \frac{L}{2} \cos \alpha) - \sin^2(k_2 \frac{L}{2} \sin \alpha) \end{aligned} \quad (18)$$

in eq. (17) and the following integral formulas<sup>8</sup>

$$\int_0^1 x^{\lambda-1} (1 + \alpha x^p)^{-\mu} (1 + \beta x^p)^{-\nu} dx = \frac{1}{p} \alpha^{-\lambda/p} B\left(\frac{\lambda}{p}, \mu + \nu - \frac{\lambda}{p}\right) {}_2F_1\left(\nu, \frac{\lambda}{p}; \mu + \nu; 1 - \frac{\beta}{\alpha}\right) \quad (19)$$

and

$$\int_0^{\infty} x^{\mu-1} \sin^2(ax) dx = - \frac{\Gamma(\mu) \cos\left(\frac{\mu\pi}{2}\right)}{2^{\mu+1} \alpha^{\mu}} \quad (20)$$

we obtain the result

$$\begin{aligned} V &= \frac{16\Gamma[(4-\nu)/2]}{(4\pi)^{(\nu+2)/2}} B\left(\frac{1}{2}, \frac{5-\nu}{2}\right) \cos(2\alpha) \frac{\Gamma(\nu-4) \cos\left(\frac{\nu-4}{2}\pi\right)}{L^{\nu-4}} \\ &\times \left\{ (\sin^2\alpha)^{(2-\nu)/2} {}_2F_1\left(1, \frac{1}{2}; \frac{6-\nu}{2}; \frac{1}{\sin^2\alpha}\right) \right. \\ &\left. + (\cos^2\alpha)^{(2-\nu)/2} {}_2F_1\left(1, \frac{1}{2}; \frac{6-\nu}{2}; \frac{1}{\cos^2\alpha}\right) \right\} \quad (21) \end{aligned}$$

Using the relations

$$B(x, y) = \frac{\Gamma(x) \Gamma(y)}{\Gamma(x+y)} \quad (22)$$

and

$$\Gamma(n+1) = \Gamma(n)$$

we can write eq. (21) as

$$\begin{aligned} V &= \frac{16\cos(2\alpha) \cos[(\nu\pi)/2] \Gamma[(5-\nu)/2] \Gamma(\nu-4)}{(4\pi)^{(\nu+1)/2} L^{\nu-4} (4-\nu)} \\ &\times \left\{ \frac{{}_2F_1\left(1, \frac{1}{2}; \frac{6-\nu}{2}; \frac{1}{\sin^2\alpha}\right)}{(\sin^2\alpha)^{(\nu-2)/2}} + \frac{{}_2F_1\left(1, \frac{1}{2}; \frac{6-\nu}{2}; \frac{1}{\cos^2\alpha}\right)}{(\cos^2\alpha)^{(\nu-2)/2}} \right\} \quad (23) \end{aligned}$$

For  $\nu=4$ , eq. (23) has a pole with residue

$$R = - \frac{\cotg(2\alpha)}{2\pi} \quad (24)$$

The above result corresponds to the case in which the arrows leave the vertex (the present case) or enter the vertex. On the other cases the sign must be changed. Note that the result in eq. (24) can be obtained by considering the residue associated with each pair of segments that compose the actual case.

To analyze the cases  $v = 3, 5, 6, \dots$ , we make use of the formulae 9.131-2 and 9.132-1 of ref. 8 and of the relation  $\Gamma(1-x)\Gamma(x) = \pi/\sin(\pi x)$ . We then obtain the final result

$$\begin{aligned}
 V = & \frac{16\cos(2\alpha)\cos[(v\pi)/2]\Gamma[(5-v)/2]\Gamma[(6-v)/2]\Gamma(v-4)}{(4\pi)^{(v+1)/2}L^{v-4}(4-v)} \\
 & \times \left\{ \frac{-\operatorname{tg}^2\alpha}{(\sin^2\alpha)^{(v-2)/2}} \frac{\Gamma(-1/2)}{\Gamma(1/2)\Gamma[(4-v)/2]} {}_2F_1\left(1, \frac{5-v}{2}; \frac{3}{2}; -\operatorname{tg}^2\alpha\right) \right. \\
 & + \frac{(-\operatorname{tg}^2\alpha)^{1/2}}{(\sin^2\alpha)^{(v-2)/2}} \frac{\Gamma[(v-3)/2] \sin[\pi/2(v-3)]}{\Gamma(1/2)} {}_2F_1\left(\frac{1}{2}, \frac{4-v}{2}; \frac{1}{2}; -\operatorname{tg}^2\alpha\right) \\
 & + \frac{\Gamma[(3-v)/2]}{(\cos^2\alpha)^{(v-2)/2}\Gamma[(4-v)/2]\Gamma[(5-v)/2]} {}_2F_1\left(1, \frac{1}{2}; \frac{v-1}{2}; -\operatorname{tg}^2\alpha\right) \\
 & \left. + \frac{(-\operatorname{tg}^2\alpha)^{(3-v)/2}}{(\cos^2\alpha)^{(v-2)/2}} \frac{\Gamma[(v-3)/2]}{\Gamma(1/2)} {}_2F_1\left(\frac{4-v}{2}; \frac{5-v}{2}; \frac{5-v}{2}; -\operatorname{tg}^2\alpha\right) \right\} \quad (25)
 \end{aligned}$$

For  $v=5$ , the RHS in eq. (25) has no pole. (The pole of  $\Gamma(\frac{5-v}{2})$  is compensated by the zero of  $\cos(\frac{v\pi}{2})$ ). For even  $v$  ( $v > 8$ ) there is a pole associated with  $\Gamma(\frac{6-v}{2})$ . For  $v=3$ ,  $V$  given by eq. (25) has one pole whose residue is

$$R = -\frac{L}{\pi} \cos(2\alpha) \quad (26)$$

Then, for  $v=3$  and 4,  $V$  given by eq. (25) has poles whose residue depends on geometric factors as the length of the segments and the angle between them.



### 3. TWO DIMENSIONAL CASE

Ir. the case of intersection of two-segments (see fig.1), for  $\nu=2$ , the analysis is simple. As we can see from eq. (25) there is no pole for such value of  $\nu$ . In the case of two segments that form an angle  $2\alpha$  (see fig.2) we proceed in different way.

As we know, being  $\Gamma(\frac{\nu-2}{2})x^{2-\nu}$  the potential in  $\nu$  dimensions, we obtain the two dimensional potential  $\ln r$  from

$$\left. \frac{d}{d\nu} (\nu-2) \Gamma\left(\frac{\nu-2}{2}\right) x^{2-\nu} \right|_{\nu=2}$$

Then, for  $\nu=2$  we take the following expression for the interaction between the segments

$$V(\nu=2) = \frac{d}{d\nu} \left[ - \frac{2(\nu-2)\Gamma[(\nu-2)/2]}{8\pi^{\nu/2}} \frac{L_2^{4-\nu}}{(4-\nu)} \cos^2\alpha (\sin\alpha)^{2-\nu} \right. \\ \times \int_{-1}^{\infty \operatorname{sgn}(\cos\alpha)} d\lambda (1 + h^2 \cot^2\alpha) (\nu-2)/2 \\ \left. \frac{(\nu-2)\Gamma[(\nu-2)/2](L_1^{4-\nu} + L_2^{4-\nu} - |\vec{L}_1 - \vec{L}_2|^{4-\nu})}{4\pi^{\nu/2} (3-\nu)(4-\nu)} \right]_{\nu=2} \quad (27)$$

The gauge term in eq. (27) is finite, as well as all the terms in which appears a derivative of a function of  $\nu$  multiplied by the integral. Consider, now, the term in which appear the derivative of the integral. For this we use the following integral formulae<sup>8</sup>

$$\int_0^1 (1 + bx^m)^{-k} dx = \sum_{k=0}^{\infty} \binom{k}{k} \frac{b^k \Gamma(1+km)}{\Gamma(2+km)}, \quad b^2 > 1 \quad (28)$$

and

$$\int_0^{\infty} x^{\lambda-1} (k + \alpha x^p)^{-\mu-\nu} dx = \frac{1}{p} \alpha^{-1/p} B\left(\frac{\lambda}{p}, \mu+\nu - \frac{\lambda}{p}\right) \quad (29)$$

With the use of eqs. (28) and (29) we obtain

$$\int_{-1}^{\infty} d\lambda (1 + \lambda^2 \cotg^2 \alpha)^{(\nu-2)/2} = \sum_{k=0}^{\infty} \binom{\frac{\nu-2}{2}}{k} \frac{(\cotg^2 \alpha)^k}{2k+1} + \frac{1}{2} \frac{1}{\cotg \alpha}$$

Taking the derivative  $d/d\nu$  of eq. (30) and putting  $\nu=2$ , we see that the result is finite, because for this value of  $\nu$  the term  $B\left(\frac{1}{2}, \frac{1-\nu}{2}\right)$  has no divergences. Therefore, for  $\nu=2$ , the interaction between these segments is finite.

Then, by joining smoothly the two segments that form an angle or that have an intersection we get a loop with cusp and intersection, respectively. As we have shown that the divergences are absent in two dimensions, in the case in which the contour has an angle or generally an intersection, we conclude that the two dimensional Wilson loop evaluated around a closed contour with a cusp or an intersection is finite. This result is expected, as we know that the Wilson loop expectation value for pure gauge theories gives a trivial area law in two dimensions, but it is a nice example of how the information for  $\nu=2$  is extracted from the general expression for any  $\nu$ .

We would like to thank Profs. C.G. Bollini and J. J. Giambiagi of the Centro Brasileiro de Pesquisas Físicas (Brazil), for useful discussions on this subject.

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#### Resumo

Discute-se a influência de cúspides e interseções no cálculo do loop de Wilson no espaço de  $\nu$ -dimensões. Em particular, para o caso bi-dimensional, mostra-se que não há divergências.