Revista Brasileira de Física, Vol. 14, nº 3, 1984

On the Classical Wilson Loop for a Class of Gauge Field Copies

VALDIR B. BEZERRA

Departamento de Física, Universidade Federal da Paraíba, 58000, João Pessoa, PB, Brasil

Recebido em 5 de Novembro de 1984

Abstract We compute ordered integrals along circles for a class of potentials that give the same field strength. We discuss the asymptotic behavior of the Wilson loop associated to these potentials.

In Abelian gauge theories, like electrodynamics, there is a simple relation between fields and potentials. Two or more potential configurations satisfying the gauge constraint give rise to the same field strength, and we say that these potentials are equivalent. Inversely, given the field çtrength configuration, all possible potential configurations are connected by gauge transformations. Therefore, in Abelian gauge theories the fields completely determine the theory.

In constrast with electrodynamics, or more generally, with the Abelian case, in non-Abelian gauge theories the situation, even at a classical level, is more complicated, and a surprising phenomenonoccurs, that is, the knowledge of the field strengths (in four space-time dimensions) does not in general determine the potentials uniquely, even up to a gauge transformation. Sometimes, for the same field strength there exists a set of potentials not gauge equivalent that generate it. The set of potentials with this property have been called *gauge field copies* or *field strength copies*.

Wu and Yang[1] have shown the first example of a field generated by potentials that are not gauge equivalent. There are other similar examples in the literature[2].

Lately, the Wilsori loop[3] has gained considerable interest in gauge theories, because it is a gauge invariant quantity and it is believed that it can act as a dynamical variable containing the exact information about the gauge fields[4]. Thus it is important to know as much

Work partially supported by CNPq (Brazilian Government Agency).

as possible about its properties. Now put the following question: is there a relation among the Wilson loops for potentials that give the same field strength? This point is very important, because it is known that the final strengths do not contain all the physical Information in the case of non-Abelian gauge theorles, and therefore, the physical similarity of the field strength copies must be required in the set of variables that are gauge invariant, such as the Wilson loops.

In this paper we give an example of potentials[5] that generate the same field strength and the same classical Wilson loop for a particular path (circle) whose center coincides with the positions of the potentials.

We show that, in order to obtain different Wilson loops for our set of gauge field copies, It is sufficient to take the positionsofthe potentials, excluding the origin, along an axis perpendicular to the circles, passing throught the origin.

Lei: us consider the followlng potentials that give the same field strerigth[5]

$$A_\mu^{(a)} = - 2ia\, \sigma_{\mu\nu} \frac{X_\nu}{X^2} \tag{1}$$

and

$$A_\mu^{(1-a)} = - 2i(1-a)\, \sigma_{\mu\nu} \frac{X_\nu}{X^2} \tag{2}$$

with $\sigma_{\mu\nu}$ satisfying the relations

$$\sigma_{ij} = \frac{1}{2} \varepsilon_{ijk}\, \sigma_k$$

and

$$\sigma_{k4} = \frac{1}{2} \sigma_k$$

where σ_k $(k=1,2,3)$ are the Pauli matrices.

After some algebraical manipulations we determine the field and currents corresponding to eqs. (1) and (2) which are respectively given by

$$F_{\mu\nu}^{(a)} = F_{\mu\nu}^{(1-a)} = \frac{4ia(1-a)}{X^4} (X_\mu\, \sigma_{\nu\rho}\, X_\rho - X_\nu\, \sigma_{\nu\rho}\, X_\rho + X^2\, \sigma_{\mu\nu}) \tag{3}$$

and

$$J_\mu^{(a)} = - J_\mu^{(1-a)} = 8ia(1-a)(1-2a) \frac{\sigma_{\mu\nu} X_\nu}{X^4} \tag{4}$$

The potentials in eqs. (1) and (2) give rise to the same field strength and equal, but opposite, currents. This corresponds to two different physical situations. Therefore, there does not exist a gauge tranrformation relating $A_\mu^{(a)}$ and $A_\mu^{(1-a)}$. In the present case it ispossible to show directly that this gauge transformation cannot exist. In fact, such a transformation should commute with $F_{\mu\nu}^{(a)}$, and therefore with the Pauli matrices. Simultaneously, such a transformation should anticommute with $J_\mu^{(a)}$ and therefore with the Pauli matrices. But since this Is not possible, then $A_\mu^{(a)}$ and $A_\mu^{(1-a)}$ cannot be related by a gauge transformat ion.

Now, let us compute the loop integral (path ordered integral), which is defined by

$$U(C) = P \exp(\oint_C A_\mu \, dx^\mu) \tag{5}$$

for the potentials considered above. For this purpose we consider loops which are complete circles in the (X_1,X_2) plane, with their centers coinciding with the positions of the potentials.

We can write the eqs. (1) and (2) as

$$A_\mu^{(b)} = 2ib \; \sigma_{\mu\nu} \frac{X_\nu}{X^2} \tag{6}$$

In this specific case, the potentials in eqs. (1) and (2) (or in eq. (6)) are localized at $X_1^0 = X_2^0 = X_3^0 = X_4^0 = 0$ and we have explicitly, with X in the (X_1,X_2)-plane, that

$$A_\mu^{(b)} = 2ib \frac{\sigma_{\mu 1}X_1 + \sigma_{\mu 2}X_2}{X^2} \tag{7}$$

where

$$X^2 = X_1^2 + X_2^2 \tag{8}$$

Now, using polar variables in the (X_1,X_2)-plane, we write

$$A_\mu^{(b)} dx^\mu = A(\theta,r)d\theta \equiv iB(\theta,r)d\theta \tag{9}$$

where

$$B(\theta,r) = - b\sigma_3 \tag{10}$$

In analogy with Bollini, Giambiagi and Tiomno[6], we obtain for the loop integral

$$U_{2\pi,0}(r) = P \exp\left[\int_0^{2\pi} B(\theta,r)d\theta\right]$$

In the case of a complete circle, the expression

$$U^{(b)}(r) = -\cos(2\pi L^{(b)}) - i\sigma_3 \operatorname{sen}(2\pi L^{(b)}) \tag{11}$$

where

$$L^{(b)} = \pm \left(\frac{1}{2} - b\right) \tag{12}$$

If we take trace of eq. (11), we get the corresponding Wilson loop

$$W^{(b)}(r) = \operatorname{Tr} U^{(b)}(r) = -2\cos(2\pi L^{(b)}) \tag{13}$$

Writting eq. (13) separately for the potentials in eq. (1) (with $b = -a$) and in eq. (2) (with $b = -(1-a)$), we have respectively

$$W^{(a)}(r) = -2\cos\left[2\pi\left(a + \frac{1}{2}\right)\right] \tag{14}$$

and

$$W^{(1-a)}(r) = -2\cos\left[2\pi\left(\frac{3}{2} - a\right)\right] \tag{15}$$

It is trivial to verify that eqs. (14) and (15) give the same result for any value of a, independently of the radius of the circles. Therefore, in this particular case the Wilson loop is the samefor gauge field copies.

Observe that the Wilson loop (eq. (14) or (15)) for a=l is equal to +2 and this corresponds to its *vacuum* value. For this reason we say that the Wilson loop does not detect or see the particles associated with the potential (1) considered. On the other hand, for $a = 1/2$, the value of the Wilson loop is -2, which is different from the vacuum value, and therefore the Wilson loop detects or sees the particles associated with the potential (1) and (2) for this value of a.

Now consider the path (complete circle) in a plane parallel to the (X_1,X_2)-plane and centre at $(0,0,z_0,t_0)$. It is easy to compute $W^{(a)}$ and $W^{(1-a)}$ in this case. For this we make use of eq. (11) of reference 6 and recall that $A = -iB$. The final results are respectively

$$W^{(a)} = -2 \cos\left[2\pi\left[\frac{1}{4} + \frac{a(a+1)r^2}{r^2+z_0^2+t_0^2}\right]^{1/2}\right] \tag{16}$$

and

$$W^{(1-a)} = -2 \cos\left[2\pi\left(\frac{1}{4} + \frac{(1-a)(2-a)r^2}{r^2+z_0^2+t_0^2}\right)^{1/2}\right] \tag{17}$$

From eqs. (16) and (17) we see that $W^{(a)}$ and $W^{(1-a)}$ are the same only for the case $a = 1/2$, for each value of r. But, in this case rhe potentlals are the same, and therefore, this result is expected. Now, if we take the limit $r\to\infty$ In eqs. (16) and (17), we get the same values for $W^{(a)}$ and $W^{(1-a)}$. In particular, for $a=1$, the Wilson loop is equal to +2, corresponding to its *vacum* value. Thus, as in theprevious case, the Wilson loop does not detect the particles associated with the potential in eq. (1) for $a=1$. For $a=1/2$, the value of the Wilson loop is -2, different from the *vacuum* value, and therefore, also in thiscase, for large loops, the Wilson loop detects or **sees** the particles associated with the considered potentials. This brief discussion on the asymptotic behavior of the Wilson loop when $|x| \to \infty$ is important because In this limit the Wilson loop can be used to detect the presence of certain types of particles. Our interest on the particular cases of $a = 1$ anda $= 1/2$ is that for $a = 1$ we have the BPST solution with $\lambda=0$ anda $= 1/2$ we have the méron[8] solution, and we know that these solutlons plus the 't Hooft Polyakov[9] monopole are the most interesting solutlons of the Yang-Mills equations.

From eqs. (16) and (17) we note that in general $W^{(a)}$ is different from $W^{(1-a)}$. Thus, it is sufficient to eliminate the coincidence between the positions of the potentials and the center of the circle to get a dlscrimination between the potentials that give hte same field strength via the Wilson loop.

It is interesting to make an attempt to clarify the results obtalned. As we know, the loop operator (loop integral) given by eq. (5), which takes on values in the gauge group, has a direct geometrical interpretation, as it represents the parallel transport of the theory. Since the Wilson loop is related to the parallel displacement of a vector along a closed path, and we have just demonstrated that, in general, the Wilson loop discriminates between gauge field copies, it is expected that the parallel displacement of a vector in the field generated by $A_\mu^{(\alpha)}$ and $A_\mu^{(1-\alpha)}$ results in different angular deviations.

To investigate this point we make use[10] of the vector representation of the rotation group instead of the spinor representation, which we have used in previous calculations.

Following reference 10, we get that the results of the calculations in this representation are that the angular deviations are the same, in the case in which the positions of the potentials coincides with the center of the circle, in agreement with the fact that the Wilson loop does not discriminate between the two potentials, as obtained previously for this situation. For a $\neq 1/2$, the angular deviations are the same only in the case of very large circles. For a circle of arbitrary radius, we have that, in general the angular deviations are different, consistently with our previous results.

Thus, in general the Wilson loop discriminates between different and non-equivalent (not related by a gauge transformation) potentials, because when we come back to the starting point of the closed curve, the angular shif'ts between the parallel transferred vector and the original one in the f'ield generated by $A_\mu^{(\alpha)}$ are different from the angular shifts in the case of $A_\mu^{(1-a)}$

Only in the particular case in which the positions of the potentials coincides with the center of the path, the Wilson loop does not distinguishes between two different physical situations. Then, there is a particular situation in which we have *copies* of the Wilson loop value.

In conclusion, we can say that in general the Wilson loop corresponding to the potentials in eqs. (1) and (2), which describe different physical situations, are distinct, and therefore it discriminates between different physical situations. This is an important point because it is expected that the Wilson loop constitutes an adequate variable to give an intrinsic and complete description of the non-Abelian gauge theories.

The author is indebted to profs. J.J.Giambiagi and C.G. Bollini for useful discussions on the subject.

REFERENCES

1. T.T.Wu and C.N.Yang, Phys. Rev. D12, 3843 (1975).
2. S.Deser and F.Wilczek, Phys. Lett. 658, 391 (1976); S.Coleman, Phys. Lett. 708, 59 (1977); C.G.Bollini, J.J.Giambiagi and J.Tiomno, Phys. Lett. 838, 185 (1979); S.Solomon, Nucl. Phys. 147B, '74 (1979).
3. K.Wilson, Phys. Rev. D10, 2445 (1974).
4. J.L.Gervais and A.Neveu, Nucl. Phys. 153 B, 445 (1979); J.L.Gervais, M.T.Jaekel and A.Neveu, Nucl. Phys. 155B, 75 (1979); A.M. Polyakov, Phys. Lett. 828, 247 (1979).
5. C.G.Bollini, J.J.Giambiagi and J.Tiomno, Rev.Bras.Fís. 9, 229 (1979).
6. C.G.Bollini, J.J.Giambiagi and J.Tiomno, Nuovo Cim. 59A, 412 (1980).
7. A.A.Belavin, A.M.Polyakov, A.S.Schwartz and Yu.S.Tyupkin, Phys.Lett. 598, 85 (1975).
8. V.de Alfonso, S.Fubini and G.Furlan, Phys. Lett. 65B, 163 (1976).
9. G. 't Hooft, Nucl. Phys. 798, 276 (1974); A.M.Polyakov, Sov. Phys. JETP 41, 988 (1975).
10. V.B. Bezerra, Rev. Brns. Fís. 12, 67 (1982).

Resumo

Calculamos Integrais ordenadas ao longo de círculos para potenciais que fornecem o mesmo tensor intensidade de campo. Discutimos o comportamento assintótico do 'loop' de Wilson associado a estes potenciais.