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Superspace in the Dirac-Kähler Framework in two Dimensions

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Abstract In an attempt to describe the rigid superspace on the Clifford algebra of differential forms we present explicitly, as an example, the two dimensional supersymmetric gauge field theory. Use is only made of Clifford product and Dirac-Kähler operator on the basis of differential forms with spinorial Loreintz transformation law.

1. INTRODUCTION

Recently, in the framework of differential geometric generalization of the Dirac equation, Becher and Joos[1] succeeded in constructing a formalism in which spinor fields can be described by forms Φ with the correct spinorial Lorentz transformation law

$$\delta\Phi = J^{\mu\nu}\,\Phi\,\delta\beta_{\mu\nu} = (x^{\mu}\partial^{\nu} - x^{\nu}\partial^{\mu} + \frac{1}{2}\,S^{\mu\nu}\vee)\;\Phi\;\delta\beta_{\mu\nu} \tag{1}$$

where $S^{\mu\nu} = dx^{\mu} \wedge dx^{\nu}$ and $\vee$ repreçentç the Clifford product.The Clifford algebra structure is introduced by an associative product for differential forms'

$$dx^{\mu} \vee dx^{\nu} = dx^{\mu} \wedge dx^{\nu} + g^{\mu\nu} \tag{2}$$

and therefore $\gamma^{\mu} \to dx^{\mu}\vee$ defines a representation of the algebra of γ-matrices in the 2^{d} dimensional space of differential forms, where d is the dirnension of space-time.

The above equivalence makes it tempting to describe the global superspace on the Clifford algebra of the differential forms.

As we shall see, this description, from our point of view, is sufficiently elegant in order to justify our attempt, and moreover we

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have the hope that it can be useful in trying to write down the supersymmetry on the lattice[2].

Relations between supersymmetry and Dirac-Kähler (D-K) forms have also been discussed in ref.3.

More specifically, we treat the example of the two-dimensional supersymmetric gauge theories[4] translating the conventional superspace to our differential form language.

Our paper is organized as follows. In sec. 2 we give a derivation of the supersymmetric Lagrangean density for the N=1 case by means of the use of differential forms associated to the fields. In sec. 3 this is extended to the Abelian gauge theory, and in sec. 4 we consider the corresponding supersymmetric non-Abelian gauge theory.

2. SUPERSPACE IN THE DIFFERENTIAL FORMS FRAMEWORK. THE $N=1$ CASE IN TWO DIMENSIONS

We first consider the N=1 two dimensional superfield which has the following zero-form structure

$$\Phi(x,\theta) = A(x) + \frac{i}{2}\bar{\theta} \vee \bar{\theta} \vee dx^{12} F(x) + \frac{i}{2}\{\bar{\psi}(x),\bar{\theta}\}_{\vee} \tag{3}$$

with

$$dx^{12} = dx^1 \wedge dx^2 \text{ and } \{A,B\}_{\vee} = A \vee B + B \vee A$$

Here we use the convention that the spinorial degrees of freedom are described by the $(0,2)$ forms

$$\theta = \theta_0 + \theta_{12}\, dx^{12}$$

and (4)

$$\psi(x) = \psi_0(x) + \psi_{12}(x) dx^{12}$$

where θ_0, θ_{12} are Grassmann parameters, and $\psi_0(x)$, $\psi_{12}(x)$ are anti-commuting fields. Moreover we make use of the notation

$$\bar{\zeta} = \zeta \vee dx^2$$
$$\tilde{\zeta} = B\zeta = dx^2 \vee \zeta \vee dx^2 \tag{5}$$

where ζ is an arbitrary $(0,2)$ form and B is an anti-automorphism[1]; $A(x)$ is a real scalar field, and $F(x)$ is a real auxiliary field (both zero forms)

Following our coriventions we propose the supersymmetry generator Q to be a (0,2) form given by

$$Q = \frac{\partial}{\partial\theta} + i\,\bar{\theta} \vee (d-\delta) = \frac{\partial}{\partial\theta_0} + \frac{\partial}{\partial\theta_{12}}\, dx^{12} + i\,\bar{\theta} \vee (d-\delta) \tag{6}$$

where (d-6) is the usual D-K operator[8]. Correspondingly we introduce the *covariant derivative*

$$D = \frac{\partial}{\partial\theta} - i\,\bar{\theta} \vee (d-\delta) \tag{7}$$

We have the properties

$$\{\frac{\partial}{\partial\theta},\theta\}_V = 2$$

$$\{\frac{\partial}{\partial\theta},\tilde{\theta}\}_V = \{\frac{\partial}{\partial\tilde{\theta}},\theta\}_V = \frac{\partial}{\partial\theta} \vee \frac{\partial}{\partial\theta} = \theta \vee \theta = 0 \tag{8}$$

$$D \vee D = - Q \vee Q = -2\, i\, dx^2 \vee (d-\delta)$$

$$D \vee \tilde{D} = \frac{\partial}{\partial\theta} \vee \frac{\partial}{\partial\bar{\theta}} - i\,\bar{\theta} \vee \frac{\partial}{\partial\theta} \vee (d-\delta) + i\,(d-\delta) \vee \bar{\theta} \vee \frac{\partial}{\partial\theta} - \bar{\theta} \vee \bar{\theta}\,(d-\delta)$$

and

$$\{D,\tilde{D}\}_V = \{Q,\tilde{Q}\}_V = 0$$

where

$$\frac{\partial}{\partial\bar{\theta}} = \frac{\partial}{\partial\theta} \vee dx^2$$

The supersymmetry transformations on Φ are generated by Q through

$$\delta\Phi = \frac{1}{2}\{\varepsilon \vee Q \vee \Phi + dx^2 \vee \varepsilon \vee Q \vee \Phi \vee dx^2\} \tag{9}$$

where ε is a constant Grassmannian (0,2) form

$$\varepsilon = \varepsilon_0 + \varepsilon_{12}\, dx^{12}$$

In components, using expressions (3) and (6) we find

$$\delta A(x) = \frac{i}{2}\{\bar{\varepsilon},\bar{\psi}(x)\}_V$$

$$\delta F(x) = \frac{i}{2}\left[(d-\delta)\psi(x),\bar{\varepsilon}\right]_V \tag{10}$$

$$\delta\psi(x) = \varepsilon F(x) + dx^{12} \vee (d-\delta)\bar{\varepsilon}A(x)$$

where

$$[A,B]_V = A \vee B - B \vee A$$

As usual the corresponding supersymmetric Lagrangean density is obtained by the last component of

$$\frac{1}{2}\,\Phi \vee D \vee \tilde{D} \vee \Phi = \ldots + \frac{\bar{\theta}\,\bar{\theta}}{2}\{F^2(x) - A(x)(d-\delta)^2 A(x)$$

$$+ \frac{i}{2}\,\bar{\psi}(x) \vee (d-\delta)\psi(x) + \frac{i}{2}\,dx^2 \vee \bar{\psi}(x) \vee (d-\delta)\psi(x) \vee dx^2\} \tag{11}$$

Noting that

$$\theta \vee \bar{\theta} = -2\,\theta_0\,\theta_{12}\,dx^{12}$$

and integrating over θ_0 and θ_{12}, the volume element dx^{12} emerges naturally and we recover the usual supersymmetric N=l Lagrangean[4,5] after using

$$(d-\delta) = dx^{\mu} \vee \partial_{\mu}$$

the mappiny

$$\gamma^{\mu} \to dx^{\mu} \vee$$

and introducing the spinor components

$$\psi_1 = \frac{\psi_0 + \psi_{12}}{\sqrt{2}}\,,\quad \psi_2 = \frac{\psi_0 - \psi_{12}}{\sqrt{2}} \tag{12}$$

with the choices $\gamma^2 = \sigma_2$ and $\gamma^1 = i\sigma_1$

3. THE SUPERSYMMETRIC ABELIAN GAUGE THEORY

In order to study the supersymmetric Abelian gauge theory we introduce the Grassmannian (0,2) form describing the vector supermultiplet

$$V(x) = \xi(x) + \frac{1}{2}\,\pi(x) \vee \theta + \frac{1}{2}\,B(x) \vee dx^{12} \vee \bar{\theta} - \frac{i}{2}\,\bar{\theta} \vee \bar{\theta} \vee \zeta(x) \tag{13}$$

where $\xi(x)$, $\zeta(x)$ are Grassmannian real (0,2) forms

$$\pi(x) = M(x) + N(x)\,dx^{12}$$

with $N(x)$, $M(x)$ real scalar fields and $B(x)$ is a vector one-form

$$B(x) = B_{\mu}(x)\,dx^{\mu} \tag{14}$$

Now let us consider the Abelian gauge transformation

$$\delta_g V = i\, dx^{12} \vee \tilde{D} \vee \Lambda(x) \tag{15}$$

where $\Lambda(x)$ is a *real* scalar superfield of the type given in eq.(3). It is easy to see from eq. (13) that $\xi(x)$ and $N(x)$ can be gauged away (*Wess-Zumino* gauge) and the gauge invariant quantities are $M(x)$, $dB(x)$ and

$$\lambda(x) = \left[- (d-\delta)\bar{\xi}(x) + \zeta(x)\right]$$

Therefore we can write

$$V(x) = \frac{M(x)}{2}\, dx^{12} \vee \theta + \frac{B(x)}{2} \vee dx^{12} \vee \bar{\theta} - \frac{i}{2}\, \bar{\theta} \vee \bar{\theta} \vee \lambda(x) \tag{16}$$

and the corresponding zero-form quantity

$$F(x) = \frac{1}{2}\, dx^{12} \vee (D \vee V(x) - dx^2 \vee D \vee V(x) \vee dx^2) \tag{17}$$

is gauge invariant

$$\delta_g F(x) = \frac{i}{2} \{D,\tilde{D}\}_V \Lambda(x) = 0$$

From eqs (16) and (17) we obtain

$$F(x) = M(x) + \frac{i}{2}\, dx^{12} \vee \{\bar{\lambda}(x),\bar{\theta}\}_V - \frac{i}{2}\, \bar{\theta} \vee \bar{\theta} \vee dB(x) \tag{18}$$

which has the form of eq. (3) and therefore by eq. (11) leads to the Lagrangean density

$$L_V(x) = \frac{1}{2}\, (dx^{12} \vee dB(x))^2 - \frac{1}{2}\, M(x)\Box M(x) + \frac{i}{4}\, \bar{\lambda}(x) \vee (d-\delta)\lambda(x)$$

$$+ \frac{i}{4}\, dx^2 \vee \bar{\lambda}(x) \vee (d-\delta)\lambda(x) \vee dx^2 \tag{19}$$

It is easy to see that

$$\frac{1}{2}\, (dx^{12} \vee dB(x))^2 = -\frac{1}{4}\, F_{\mu\nu}(x) F^{\mu\nu}(x) \tag{20}$$

where

$$F_{\mu\nu}(x) = \partial_\mu B_\nu(x) - \partial_\nu B_\mu(x)$$

Now let us consider eq. (3) with $A(x)$, $F(x)$ and $\psi(x)$ ascomplex fields. We are led to introduce also the supefield

$$\Phi^*(x) = A^*(x) + \frac{i}{2}\,\bar{\theta} \vee \bar{\theta} \vee dx^{12}\, F^*(x) + \frac{i}{2}\,\{\bar{\psi}^*(x), \bar{\theta}\}_{\vee} \vee dx^{12} \tag{21}$$

and we assume that $\Phi(x)$ and $\Phi^*(x)$ describe the matter field with the gauge transformations properties

$$\begin{aligned} \delta\Phi(x) &= -\,ig\,\Lambda(x) \vee \Phi(x) \\ \delta\Phi^*(x) &= ig\,\Lambda(x) \vee \Phi^*(x) \end{aligned} \tag{21'}$$

$\Lambda(x)$ is a real scalar superfield. Imposing eq.(15) we find that the *covariant derivatives* for $\Phi(x)$ and $\Phi^*(x)$ are given respectively by

$$\nabla = D - g\,dz^{12} \vee \tilde{V}$$

and

$$\vec{\nabla} = \tilde{D} - g\,dx^{12} \vee V \tag{22}$$

with the properties

$$\delta_g(V \vee \Phi(x)) = -\,ig\,\Lambda(x) \vee V \vee \Phi(x)$$

and

$$\delta_g(\vec{\nabla} \vee \Phi^*(x)) = ig\,A(x) \vee \vec{V} \vee \Phi^*(x) \tag{23}$$

This leads naturally to the gauge invariant quantity

$$P_0\{((\nabla \vee \Phi(x)) \vee (\vec{\nabla} \vee \Phi^*(x))) \vee dx^{12}\} \tag{24}$$

where P_0 projects on zero forms. The corresponding action is

$$S_M = -\frac{1}{2}\int d\bar{\theta} \vee d\bar{\theta}\, P_0\{((\nabla \vee \Phi(x)) \vee (\vec{\nabla} \vee \Phi^*(x))) \vee dx^{12}\} \tag{25}$$

and therefore the Lagrangean density is written as

$$\begin{aligned} L_M = -\frac{1}{2}\,P_0\,\{&(D \vee \Phi) \vee (\tilde{D} \vee \Phi^*) + g^2(\tilde{V} \vee \Phi) \vee (V \vee \Phi^*) \\ &- g\,dx^{12}\,[(\tilde{V} \vee \Phi) \vee (\tilde{D} \vee \Phi^*) + (D \vee \Phi) \vee (V \vee \Phi^*)]\} \end{aligned} \tag{26}$$

Therefore introducing eqs. (3), (7), (16) and *(21)* in to eq. (26), we obtain

$$L_M(x) = \frac{1}{2} F(x)F^*(x) + \frac{i}{2} P_0\left[\bar{\psi}^*(x) \vee (d-\delta)\psi(x)\right] + \frac{1}{2}(d-\delta)A(x) \vee (d-\delta)A^*(x)$$

$$- \frac{g^2}{2} A(x)A^*(x) \ \left|M^2(x) - B(x) \vee B(x)\right] - \frac{gM(x)}{4} P_0\left[\bar{\psi}^*(x) \vee \bar{\psi}(x)\right]$$

$$+ \frac{ig}{4} P_0\left[B(x)A^*(x) \vee (d-\delta)A(x) - A(x)((d-\delta)A^*(x)) \vee B(x)\right]$$

$$- \frac{g}{4} P_0\left[\bar{\psi}(x) \vee B(x) \vee \psi^*(x)\right] - \frac{g}{4} P_0\{A(x)\bar{\psi}^*(x) \vee \bar{\lambda}(x)$$

$$+ A^*(x)\bar{\lambda}(x) \vee \bar{\psi}(x)\} \tag{27}$$

which, in terms of spinor components, uslng eq.(12), with $\gamma^2 = a_{,,}$ $\gamma^1 =$ $= -i\sigma_1$ and $\gamma^5 = \gamma^1\gamma^2 = \sigma_3$ gives for the total Lagrangean density the expresslon

$$L(x) = L_V(x) + L_M(x) = -\frac{1}{4} F_{\mu\nu}(x)F^{\mu\nu}(x) + \frac{i}{2} \bar{\lambda}(x)\gamma^\mu \partial_\mu \lambda(x)$$

$$- \frac{1}{2} M(x)\Box M(x) + \frac{1}{2} F(x)F^*(x) + \frac{i}{2} \bar{\psi}(x)\gamma^\mu\partial_\mu \psi(x)$$

$$+ \frac{1}{2} \partial_\mu A(x)\partial^\mu A^*(x) - \frac{g^2}{8} A^*(x)A(x) \left[M^2(x) - B_\mu(x)B^\mu(x)\right]$$

$$+ \frac{ig}{4} M(x)\bar{\psi}(x)\gamma^5\psi(x) - \frac{ig}{4} B_\mu(x) \left(A(x)\overleftrightarrow{\partial}^\mu A^*(x)\right)$$

$$+ \frac{g}{4} \bar{\psi}(x)\gamma^\mu\psi(x)B_\mu(x) + \frac{ig}{4} A^*(x)\bar{\lambda}(x)\gamma^5 \psi(x)$$

$$+ \frac{ig}{4} A(x)\bar{\psi}(x)\gamma^5 \psi(x) \tag{28}$$

wlth

$$\bar{\psi} = \psi^+ \gamma^2 \quad , \quad \psi_1^+ = \frac{\psi_0^* + \psi_{12}^*}{\sqrt{2}} \text{ , etc..}$$

4. THE SUPERSYMMETRIC NON-ABELIAN GAUGE THEORY

Let us now generalize the previous treatment to the case of non-Abelian gauge theory. For this purpose we start by defining the non -Abelian gauge transformation for the spinor superfield

$$\delta_g V = i dx^{12} \vee \tilde{D} \vee \Lambda(x) \vee g\left[\Lambda(x), V(x)\right]_V \tag{29}$$

Correspondingly we introduce the covariant zero form F given by

$$F = \frac{1}{2}\, dx^{12} \vee (D \vee V - \tilde{D} \vee V) - \frac{ig}{2}\{\tilde{V}, V\}_{\vee} \tag{30}$$

which under *(29)* transforms like

$$\delta_g\, F(x) = g\left[\Lambda(x),\ F(x)\right]_{\vee} \tag{31}$$

We can define a *covariant derivative*

$$\vec{\nabla} \vee F = D \vee F - ig\, dx^{12} \vee \left[\tilde{V}, F\right]_{\vee} \tag{32}$$

with the transformation property

$$\delta_g(\vec{\nabla} \vee F) = g\left[\Lambda(x),\ \vec{\nabla} \vee F\right]_{\vee} \tag{33}$$

It is also useful to introduce the other *covariant derivative* $\tilde{\vec{\nabla}}$ defined by

$$\tilde{\vec{\nabla}} \vee F = \tilde{D} \vee F + ig\, dx^{12} \vee \left[V(x), F(x)\right] \tag{34}$$

which under a gauge transformations gives

$$\delta_g(\tilde{\vec{\nabla}} \vee F) = g\left[\Lambda(x),\ \tilde{\vec{\nabla}} \vee F\right]_{\vee} \tag{35}$$

The supersymmetric pure Yang-Mills action is written

$$S_{Y-M} = \frac{1}{2}\int d\bar{\theta} \vee d\bar{\theta}\ P_0\{(\tilde{\vec{\nabla}} \vee F(x)) \vee (\tilde{\vec{\nabla}} \vee F(x)) \vee dx^{12}\} \tag{36}$$

From eq. *(30)* we obtain $F(x)$ in components

$$F(x) = M(x) + \frac{i}{2}\, dx^{12} \vee \{\bar{\lambda}(x), \bar{\theta}\}_{\vee} - \frac{i}{2}\, \bar{\theta} \vee \bar{\theta} \vee dB(x)$$

$$- \frac{ig}{4}\, B_{\mu}(x) B_{\nu}(x)\, dx^{\mu\nu} \vee \bar{\theta} \vee \bar{\theta} \tag{37}$$

where all fields are SU_n matrices. Then eq.(36), after some algebra, gives for tke Lagrangean density

$$L(x) = \frac{1}{2} P_0\{ i\, \lambda(x) \vee (dx^2 \vee (d-\delta) + \frac{g}{2} [dx^2 \vee B(x), \ ]_V)\lambda(x)$$

$$+ ((d-\delta) + \frac{g}{2} [B(x), \ ]_V)\, M(x) \vee ((d-\delta) + \frac{g}{2} [B(x), \ ]_V)\, M(x)$$

$$- \frac{1}{2} F_{\mu\nu}(x)\, F^{\mu\nu}(x) - \frac{ig}{2} \bar{\lambda}(x) \vee [\lambda(x), M(x)]_V \vee dx^2 \} \tag{38}$$

where

$$F_{12}(x) = (\partial_1 B_2(x) - \partial_2 B_1(x)) + \frac{g}{2} [B_1, B_2] \tag{39}$$

Finally let us mentlon that the following Leibnitz rules canbe proven for D as given by eq.(7)

$$D \vee (\Lambda_1(x) \vee \Lambda_2(x)) = (D \vee \Lambda_1(x)) \vee \Lambda_2(x) + \Lambda_1(x) \vee (D \vee \Lambda_2(x))$$

$$D \vee (V \vee \Lambda(x)) = (D \vee V(x)) \vee \Lambda(x) - V(x) \vee (D \vee \Lambda(x))$$

where Λ_1, Λ_2 and A are zero forms and $V(x)$ is a Grassmannian (0,2) form.

In conclusion we have succeeded in deriving thetwodimensional supersymmetric gauge theories wrltten now in termsofdifferential forms. As we have already mentioned, thls can be useful in translating the theory on the lattice. A first attempt in this direction has been made for the two dimensional N=2 Wess-Zumlno mode1[6]. By a convenient choice of the differential forms assoclated to the fields, it is possible to extend the procedure to higher dimensions.

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Resumo

Numa tentativa de descrever o superespaço rígido sobre a álgebra de Clifford das formas diferenciais, apresentamos explicitamente, como exemplo, a teoria de campos de *gauge* bidimensional supersimétrica. Fazemos uso do produto de Clifford e do operador de Dirac-Kähler na base das formas diferenciais com leis de transformação de Lorentz e spinoriais.