Superspace in the Dirac-Kähler Framework in two Dimensions

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Received on 26 December 1984

Abstract In an attempt to describe the rigid superspace on the Clifford algebra of differential forms we present explicitly, as an example, the two dimensional supersymmetric gauge field theory. Use is only made of Clifford product and Dirac-Kähler operator on the basis of differential forms with spinorial Lorentz transformation law.

1. INTRODUCTION

Recently, in the framework of differential geometric generalization of the Dirac equation, Becher and Joos\(^1\) succeeded in constructing a formalism in which spinor fields can be described by forms \(\Phi\) with the correct spinorial Lorentz transformation law

\[
\delta \Phi = \tau^{\mu\nu} \Phi \delta \beta_{\mu\nu} = (\gamma^{\mu}_{\nu} - \gamma^{\nu}_{\mu} + \frac{1}{2} \gamma^{\mu\nu}) \Phi \delta \beta_{\mu\nu}
\]

(1)

where \(\tau^{\mu\nu} = \partial \gamma_{\mu} \wedge \partial \gamma_{\nu}\) and \(\nu\) represents the Clifford product. The Clifford algebra structure is introduced by an associative product for differential forms

\[
\partial \gamma_{\mu} \wedge \partial \gamma_{\nu} = \partial \gamma_{\mu} \wedge \partial \gamma_{\nu} + \gamma^{\mu\nu}
\]

(2)

and therefore \(\gamma_{\mu} \rightarrow \partial \gamma_{\mu}\) defines a representation of the algebra of \(\gamma\)-matrices in the \(2^d\) dimensional space of differential forms, where \(d\) is the dimension of space-time.

The above equivalence makes it tempting to describe the global superspace on the Clifford algebra of the differential forms.

As we shall see, this description, from our point of view, is sufficiently elegant in order to justify our attempt, and moreover we

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*Work partially supported by FAPESP (Brazilian Government Agency).
**With a fellowship of FAPESP (Brazilian Government Agency).
***Work partially supported by FINEP and CNPq (Brazilian Government Agencies).
have the hope that it can be useful in trying to write down the supersymmetry on the lattice. Relations between supersymmetry and Dirac-Kähler (D-K) forms have also been discussed in ref. 3.

More specifically, we treat the example of the two-dimensional supersymmetric gauge theories translating the conventional superspace to our differential form language.

Our paper is organized as follows. In sec. 2 we give a derivation of the supersymmetric Lagrangean density for the N=1 case by means of the use of differential forms associated to the fields. In sec. 3 this is extended to the Abelian gauge theory, and in sec. 4 we consider the corresponding supersymmetric non-Abelian gauge theory.

2. SUPERSPACE IN THE DIFFERENTIAL FORMS FRAMEWORK. THE N=1 CASE IN TWO DIMENSIONS

We first consider the N=1 two dimensional superfield which has the following zero-form structure

$$\phi(x, \theta) = A(x) + \frac{i}{2} \bar{\theta} \gamma \gamma V \partial x^{12} F(x) + \frac{i}{2} \{\bar{\psi}(x), \theta\} V$$

with

$$\partial x^{12} = \partial x^1 \wedge \partial x^2 \text{ and } \{A, B\} V = AVB + BVA$$

Here we use the convention that the spinorial degrees of freedom are described by the (0,2) forms

$$\theta = \theta_0 + \theta_{12} \partial x^{12}$$

and

$$\psi(x) = \psi_0(x) + \psi_{12}(x) \partial x^{12}$$

where $\theta_0$, $\theta_{12}$ are Grassmann parameters, and $\psi_0(x)$, $\psi_{12}(x)$ are anti-commuting fields. Moreover we make use of the notation

$$\zeta = \zeta \gamma \gamma \partial x^2$$

$$\bar{\zeta} = B \zeta = \partial x^2 \gamma \gamma V \zeta \gamma \gamma \partial x^2$$

where $\zeta$ is an arbitrary (0,2) form and $B$ is an anti-automorphism; $A(x)$ is a real scalar field, and $F(x)$ is a real auxiliary field (both zero forms).
Following our conventions we propose the supersymmetry generator $Q$ to be a $(0,2)$ form given by

$$Q = \frac{\partial}{\partial \bar{\theta}} + i \bar{\theta} V (d-\delta) = \frac{\partial}{\partial \bar{\theta}} + \frac{\partial}{\partial \theta_{12}} dx^{12} + i \bar{\theta} V (d-\delta)$$  \hspace{1cm} (6)

where $(d-\delta)$ is the usual D-K operator. Correspondingly we introduce the covariant derivative

$$D = \frac{\partial}{\partial \bar{\theta}} - i \bar{\theta} V (d-\delta)$$  \hspace{1cm} (7)

We have the properties

$$\{\frac{\partial}{\partial \bar{\theta}}, 0\}_V = 2$$

$$\{\frac{\partial}{\partial \bar{\theta}}, \bar{\theta}\}_V = \frac{\partial}{\partial \bar{\theta}} V \frac{\partial}{\partial \theta} = \theta V \theta = 0$$  \hspace{1cm} (8)

$$D V D = - Q V Q = -2 i dx^2 V (d-\delta)$$

$$D V \bar{D} = \frac{\partial}{\partial \bar{\theta}} V \frac{\partial}{\partial \bar{\theta}} - i \bar{\theta} V \frac{\partial}{\partial \bar{\theta}} (d-\delta) + i (d-\delta) V \bar{\theta} V \frac{\partial}{\partial \theta} - \bar{\theta} V \bar{\theta} (d-\delta)$$

and

$$\{D, \bar{D}\}_V = \{Q, \bar{Q}\}_V = 0$$

where

$$\frac{\partial}{\partial \bar{\theta}} = \frac{\partial}{\partial \bar{\theta}} V dx^2$$

The supersymmetry transformations on $\Phi$ are generated by $Q$ through

$$\delta \Phi = \frac{1}{2} \{\epsilon V Q V \Phi + dx^2 V \epsilon V Q V \Phi + dx^2\}$$  \hspace{1cm} (9)

where $\epsilon$ is a constant Grassmannian $(0,2)$ form

$$\epsilon = \epsilon_1 + \epsilon_{12} dx^{12}$$

In components, using expressions (3) and (6) we find

$$\delta A(x) = \frac{i}{2} \{\epsilon, \bar{\psi}(x)\}_V$$

$$\delta F(x) = \frac{i}{2} \{ (d-\delta) \psi(x), \bar{\epsilon}\}_V$$  \hspace{1cm} (10)

$$\delta \psi(x) = \epsilon F(x) + dx^{12} V (d-\delta) \bar{\epsilon} A(x)$$
where

$$[A, B]_{\psi} = A \psi B - B \psi A$$

As usual the corresponding supersymmetric Lagrangian density is obtained by the last component of

$$\frac{1}{2} \phi \psi \bar{\psi} \mu \psi = \ldots + \frac{\theta \bar{\theta}}{2} \{ F^2 (x) - A (x) (\partial \psi)^2 A (x)$$

$$+ \frac{i}{2} \bar{\psi} (x) \psi (d-\delta) \psi (x) + \frac{i}{2} \psi (x) \bar{\psi} (x) \partial (d-\delta) \theta (x) \psi \partial x^2 \}.$$  \hfill (11)

Noting that

$$\theta \psi \bar{\theta} = -2 \theta \theta \partial x^{12}$$

and integrating over \(\theta_0\) and \(\theta_{12}\), the volume element \(\partial x^{12}\) emerges naturally and we recover the usual supersymmetric N=1 Lagrangian after using

$$(d-\delta) = \partial x^\mu \gamma^\mu,$$

the mapping

$$\gamma^\mu = \partial x^\mu \psi$$

and introducing the spinor components

$$\psi_1 = \frac{\psi_0 + \psi_{12}}{\sqrt{2}}, \quad \psi_2 = \frac{\psi_0 - \psi_{12}}{\sqrt{2}}$$  \hfill (12)

with the choices \(\gamma^2 = \sigma_2\) and \(\gamma^1 = \sigma_1\).

### 3. THE SUPERSYMMETRIC ABELIAN GAUGE THEORY

In order to study the supersymmetric Abelian gauge theory we introduce the Grassmannian \((0,2)\) form describing the vector supermultiplet

$$\nu (x) = \xi (x) + \frac{1}{2} \pi (x) \psi \theta + \frac{1}{2} B (x) \psi \partial x^{12} \psi - \frac{i}{2} \gamma \psi \partial \psi \zeta (x)$$  \hfill (13)

where \(\xi (x)\), \(\zeta (x)\) are Grassmannian real \((0,2)\) forms

$$\pi (x) = M (x) + N (x) \partial x^{12}$$

with \(N(x), M(x)\) real scalar fields and \(B(x)\) is a vector one-form

$$B (x) = B_\mu (x) \partial x^\mu$$  \hfill (14)
Now let us consider the Abelian gauge transformation

$$\delta_g V = i \, dx^{12} \, \bar{D} \, \Lambda(x)$$

(15)

where $\Lambda(x)$ is a real scalar superfield of the type given in eq. (3). It is easy to see from eq. (13) that $\xi(x)$ and $N(x)$ can be gauged away (Wess-Zumino gauge) and the gauge invariant quantities are $M(x)$, $dB(x)$ and

$$\lambda(x) = [\phi - (d-\delta) \bar{\xi}(x) + \xi(x)]$$

Therefore we can write

$$V(x) = \frac{M(x)}{2} \, dx^{12} \, \bar{\theta} + \frac{B(x)}{2} \, dx^{12} \, \bar{\theta} - \frac{i}{2} \, \bar{\theta} \, \bar{\theta} \, V \, \lambda(x)$$

(16)

and the corresponding zero-form quantity

$$F(x) = \frac{1}{2} \, dx^{12} \, V \{ D \, V \, U(x) - \bar{D} \, \bar{V} \, V \}$$

(17)

is gauge invariant

$$F(x) = \frac{i}{2} \{ D, \bar{D} \} \, V \, \lambda(x) = 0$$

From eqs (16) and (17) we obtain

$$F(x) = M(x) + \frac{i}{2} \, dx^{12} \, V \{ \bar{\lambda}(x), \bar{\theta} \} \, V \, \frac{i}{2} \, \bar{\theta} \, \bar{\theta} \, V \, dB(x)$$

(18)

which has the form of eq. (3) and therefore by eq. (11) leads to the Lagrangean density

$$L_{\omega}(x) = \frac{1}{2} \, (dx^{12} \, dB(x))^2 - \frac{1}{2} \, M(x) \, \square M(x) + \frac{i}{4} \, \bar{\lambda}(x) \, V \, (d-\delta) \lambda(x)$$

$$+ \frac{i}{4} \, dx^{12} \, V \, \bar{\lambda}(x) \, V \, (d-\delta) \lambda(x) \, V \, dx^{12}$$

(19)

It is easy to see that

$$\frac{1}{2} \, (dx^{12} \, dB(x))^2 = - \frac{1}{4} \, F_{\mu \nu}(x) \epsilon^{\mu \nu}(x)$$

(20)

where

$$F_{\mu \nu}(x) = \partial_\mu B_\nu(x) - \partial_\nu B_\mu(x)$$
Now let us consider eq. (3) with $A(x)$, $F(x)$ and $\psi(x)$ as complex fields. We are led to introduce also the superfield

$$\Phi(x) = A(x) + \frac{i}{2} \bar{\theta} \mathbf{V} \bar{\theta} \mathbf{V} \ dx^{12}$$

and we assume that $\Phi(x)$ and $\Phi^*(x)$ describe the matter field with the gauge transformations properties

$$\delta \Phi(x) = -ig \Lambda(x) \mathbf{V} \Phi(x)$$

$$\delta \Phi^*(x) = ig \Lambda(x) \mathbf{V} \Phi^*(x)$$

$\Lambda(x)$ is a real scalar superfield. Imposing eq. (15) we find that the covariant derivatives for $\Phi(x)$ and $\Phi^*(x)$ are given respectively by

$$\nabla = D - g \ dx^{12} \mathbf{V} \bar{\theta}$$

and

$$\bar{\nabla} = \bar{D} - g \ dx^{12} \mathbf{V} \theta$$

with the properties

$$\delta_g(\nabla \mathbf{V} \Phi(x)) = -ig \Lambda(x) \mathbf{V} \mathbf{V} \Phi(x)$$

and

$$\delta_g(\bar{\nabla} \mathbf{V} \Phi^*(x)) = ig \Lambda(x) \bar{\theta} \mathbf{V} \Phi^*(x)$$

This leads naturally to the gauge invariant quantity

$$P_0 \{((\nabla \mathbf{V} \Phi(x)) \mathbf{V} (\bar{\nabla} \mathbf{V} \Phi^*(x))) \mathbf{V} \ dx^{12}\}$$

where $P_0$ projects on zero forms. The corresponding action is

$$S_M = -\frac{1}{2} \int d\bar{\theta} \mathbf{V} d\theta P_0 \{((\nabla \mathbf{V} \Phi(x)) \mathbf{V} (\bar{\nabla} \mathbf{V} \Phi^*(x))) \mathbf{V} \ dx^{12}\}$$

and therefore the Lagrangean density is written as

$$L_M = -\frac{1}{2} P_0 \{((\nabla \mathbf{V} \Phi) \mathbf{V} (\bar{\nabla} \mathbf{V} \Phi^*)) + g^2(\bar{\theta} \mathbf{V} \Phi) \mathbf{V} (\theta \mathbf{V} \Phi^*) \}

- g \ dx^{12} [(\bar{\nabla} \mathbf{V} \Phi) \mathbf{V} (\bar{D} \mathbf{V} \Phi^*) + (\bar{D} \mathbf{V} \Phi) \mathbf{V} (\mathbf{V} \mathbf{V} \Phi^*)]$$

Therefore introducing eqs. (3), (7), (16) and (21) in to eq. (26), we obtain
\[ L_M(x) = \frac{1}{2} F(x) F^*(x) + \frac{i}{2} \, P_0 \left[ \bar{\psi}^*(x) \Gamma (d-\delta) \psi(x) \right] + \frac{1}{2} (d-\delta) A(x) \Gamma (d-\delta) A^*(x) \]

\[- \frac{\alpha_0^2}{4} A(x) A^*(x) \left[ M^2(x) - B(x) \Gamma B(x) \right] - \frac{\alpha_0^2}{4} \, P_0 \left[ \bar{\psi}^*(x) \Gamma \bar{\psi}(x) \right] \]

\[+ \frac{i \alpha_0}{4} \, P_0 \left[ B(x) A^*(x) \Gamma (d-\delta) A(x) - A(x) \left( (d-\delta) A^*(x) \Gamma B(x) \right) \right] \]

\[- \frac{\alpha_0^2}{4} \, P_0 \left[ \bar{\psi}(x) \Gamma B(x) \Gamma \psi^*(x) \right] - \frac{\alpha_0^2}{4} \, P_0 \left[ A(x) \bar{\psi}^*(x) \Gamma \bar{\lambda}(x) \right] \]

\[+ A^*(x) \bar{\lambda}(x) \Gamma \bar{\psi}(x) \} \quad (27) \]

which, in terms of spinor components, using eq. (12), with \( \gamma^2 = a, \quad \gamma^1 = -i \sigma_1 \) and \( \gamma^5 = \gamma^1 \gamma^2 = \sigma_3 \) gives for the total Lagrangean density the expression

\[ L(x) = L_V(x) + L_M(x) = -\frac{1}{4} F_{\mu \nu}(x) F^{\mu \nu}(x) + \frac{i}{2} \bar{\lambda}(x) \gamma^\mu \partial_\mu \lambda(x) \]

\[- \frac{1}{2} M(x) \Box M(x) + \frac{1}{2} F(x) F^*(x) + \frac{i}{2} \bar{\psi}(x) \gamma^\mu \partial_\mu \psi(x) \]

\[+ \frac{1}{2} \partial_\mu A(x) \delta^{\mu \nu} A^*(x) - \frac{\alpha_0^2}{8} A^*(x) A(x) \left[ M^2(x) - B^\mu(x) B^\mu(x) \right] \]

\[+ \frac{i \alpha_0}{2} M(x) \bar{\psi}(x) \gamma^5 \psi(x) - \frac{i \alpha_0}{4} \bar{B}_\mu(x) \left( A(x) \gamma^{\mu \nu} A^*(x) \right) \]

\[+ \frac{\alpha_0^2}{4} \bar{\psi}(x) \gamma^\mu \psi(x) B^\mu(x) + \frac{\alpha_0^2}{4} A^*(x) \bar{\lambda}(x) \gamma^5 \psi(x) \]

\[+ \frac{i \alpha_0}{4} A(x) \bar{\psi}(x) \gamma^5 \psi(x) \] \quad (28)

with

\[ \bar{\psi} = \psi^+ \gamma^2, \quad \psi^+ = \frac{\psi^*_0 + \psi^*_2}{\sqrt{2}}, \text{ etc...} \]

**4. THE SUPERSYMMETRIC NON-ABELIAN GAUGE THEORY**

Let us now generalize the previous treatment to the case of non-Abelian gauge theory. For this purpose we start by defining the non-Abelian gauge transformation for the spinor superfield

\[ \delta_\gamma \psi = \gamma^2 \psi \quad \rightarrow \quad \psi^+ = \frac{\psi^*_0 + \psi^*_2}{\sqrt{2}}, \text{ etc...} \]

\[ \delta_\gamma \psi = i dx^1 \ V \bar{D} \ V \Lambda(x) \ V g [\Lambda(x), \psi(x)] \] \quad (29)
Correspondingly we introduce the covariant zero form \( F \) given by

\[
F = \frac{1}{2} \, dx^{12} \, V \left( D \, V \, V - \bar{\sigma} \, V \, V \right) - \frac{i g}{2} \, \{ \bar{V}, V \} \tag{30}
\]

which under (29) transforms like

\[
\delta_g \, F(x) = g \left[ \Lambda(x), \, F(x) \right] \tag{31}
\]

We can define a covariant derivative

\[
\bar{D} \, V \, F = D \, V \, F - i g \, dx^{12} \, V \left[ \bar{V}, F \right] \tag{32}
\]

with the transformation property

\[
\delta_g \, (\bar{D} \, V \, F) = g \left[ \Lambda(x), \, \bar{D} \, V \, F \right] \tag{33}
\]

It is also useful to introduce the other covariant derivative \( \bar{\bar{D}} \) defined by

\[
\bar{\bar{D}} \, V \, F = \bar{\sigma} \, V \, F + i g \, dx^{12} \, V \left[ V(x), F(x) \right] \tag{34}
\]

which under a gauge transformations gives

\[
\delta_g \, (\bar{\bar{D}} \, V \, F) = g \left[ \Lambda(x), \, \bar{\bar{D}} \, V \, F \right] \tag{35}
\]

The supersymmetric pure Yang-Mills action is written

\[
S_{Y-M} = \frac{1}{2} \int d\bar{\sigma} \, V \, d\bar{\sigma} \, P_0 \left( \left( \bar{D} \, V \, F(x) \right) \, V \left( \bar{D} \, V \, F(x) \right) \, dx^{12} \right) \tag{36}
\]

From eq. (30) we obtain \( F(x) \) in components

\[
F(x) = M(x) + \frac{i}{2} \, dx^{12} \, V \left( \bar{\chi}(x), \bar{\theta} \right) - \frac{i}{2} \, \bar{\theta} \, V \, \bar{\theta} \, dB(x)
\]

\[
- \frac{i g}{4} \, B_{\mu}(x) B_{\nu}(x) \, dx^{\mu \nu} \, V \, \bar{\theta} \, V \, \bar{\theta} \tag{37}
\]

where all fields are SU\(_n\) matrices. Then eq. (36), after some algebra, gives for the Lagrangean density

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Finally let us mention that the following Leibnitz rules can be proven for $D$ as given by eq. (7)

$$D V (\Lambda_1(x) V \Lambda_2(x)) = (D V \Lambda_1(x)) V \Lambda_2(x) + \Lambda_1(x) V (D V \Lambda_2(x))$$

$$D V (V V \Lambda(x)) = (D V V(x)) V \Lambda(x) - V(x) V (D V \Lambda(x))$$

where $\Lambda_1$, $\Lambda_2$ and $\Lambda$ are zero forms and $V(x)$ is a Grassmannian $(0,2)$ form.

In conclusion we have succeeded in deriving the two-dimensional supersymmetric gauge theories written now in terms of differential forms. As we have already mentioned, this can be useful in translating the theory on the lattice. A first attempt in this direction has been made for the two dimensional $N=2$ Wess-Zumino model. By a convenient choice of the differential forms associated to the fields, it is possible to extend the procedure to higher dimensions.

REFERENCES


Resumo

Numa tentativa de descrever o superespaço rígido sobre a álgebra de Clifford das formas diferenciais, apresentamos explicitamente, como exemplo, a teoria de campos de gauge bidimensional supersimétrica. Fazemos uso do produto de Clifford e do operador de Dirac-Kähler na base das formas diferenciais com leis de transformação de Lorentz espinoriais.