Revista Brasileira de Física, Vol. 14, nº 3, 1984

Vacuum Polarization and Renormalized Charge in ν-Dimensions

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Recebido em 09 de novembro de 1984

Abstract The expression for the vacuum polarization is obtained for any momentum transfer in ν dimensions. Using the Wilson loop for QED, the renormalized electric charge in ν dimensions is calculated.

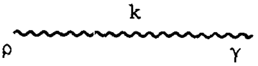
1. INTRODUCTION

It is well known that the vacuum polarization can be obtained solving the parametric integral[1]. However, the hypergeometric function that appears in the final expression diverges for large momentum transfer. This behaviour is not desirable because the vacuum polarization for large momentum transfer is important for physically observable phenomena such as the Uehling effect.

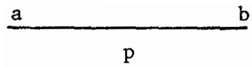
In this work, we derive an expression for the vacuum polarization which is valid for any momentum transfer. Our technique is based upon an analytic continuation of the hypergeometric function. Using the dimensional regularization[1,3] and expanding the result around $\nu = 4$ and $k^2 = 0$, the expression for the vacuum polarization is derived in a form that can be readily compared with the one obtained via cut-off regularization. The observable physical charge or renormalized charge[4] is obtained through the Wilson loop for QED; in particular, an expression for the renormalized charge in ν dimensions is derived.

2. FEYNMAN RULES AND THE LOOP AVERAGE

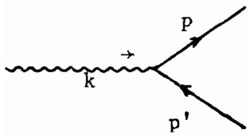
In this section, are listed for completeness the Feynman rules in Euclidian space with the metric (1,1,1,1) in ν dimensions for QED. Using the proper definition[5], one sees that the rules can be written as



$$= D_{\rho\gamma}(k) = \left(\frac{\delta_{\rho\gamma}}{k^2} - (1-\alpha)\,\frac{k_\rho k_\gamma}{k^4}\right) \tag{1}$$



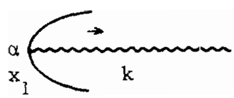
$$= \Delta_F(p) = \frac{(-i\gamma . p + m)}{p^2 + m^2} \tag{2}$$



$$= ie\gamma_\rho \; \delta(-p + p' + k) \tag{3}$$

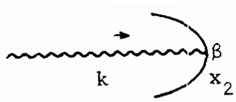
As usual, one attaches to each vertex the factor $(2\pi)^\nu$ and integrates over all internal momenta with $\int d^\nu q/(2\pi)^\nu$.

The Wilson loop vertices for QED are written as



$$= ie \oint dx_1^\alpha \, e^{-ikx_1} \tag{4}$$

and



$$= ie \oint dx_2^\beta \, e^{ikx_2} \tag{5}$$

The utility of the given rules is evident since the loop average for an Abelian theory is given by the perturbative series[6]

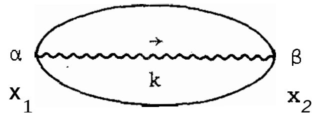
$$W(c) = \sum_n W_n(c) = 1 + \sum_{n=2}^{\infty} (ie)^n \frac{1}{n} \oint dx_1^{\alpha_1} \ldots \oint dx_n^{\alpha_n} \langle A_{\alpha_1}(x_1) \ldots A_{\alpha_n}(x_n) \rangle \tag{6}$$

where

$$\langle A_{\alpha_1}(x_1) \ldots A_{\alpha_n}(x_n) \rangle \equiv D_{\alpha_1 \ldots \alpha_n}(x_1, \ldots, x_n)$$

are Green's functions.

To second order, only the graph

$W_2 =$ 

(7)

contributes to eq. (6). Using the Feynman rules, one has

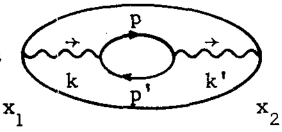
$$W_2 = -\frac{e^2}{2} \oint dx_1^\alpha \oint dx_2^\beta \int \frac{d^\nu k}{(2\pi)^\nu} e^{ik(x_2 - x_1)} \left(\frac{\delta_{\alpha\beta}}{k^2} - (1-\alpha) \frac{k_\alpha k_\beta}{k^4} \right) \tag{8}$$

It is worth-recalling that this integral is gauge independent, since

$$\oint dx_1^\alpha \, k_\alpha \, e^{-k.x_1} = 0 \tag{9}$$

The gauge depend term in eq. (8) will be dropped from now on.

The correction due to the insertion of the photon self-energy is given by

$\bar{W}_2 = \alpha$  $\beta = -\frac{e^2}{2}\Bigg\{ \frac{e^2}{(2\pi)^\nu} \oint dx_1^\alpha \oint dx_2^\beta$

$$\int \frac{d^\nu k}{(2\pi)^\nu} \int \frac{d^\nu k'}{(2\pi)^\nu} \int d^\nu p \int d^\nu p' \, e^{ik'x_2 - ikx_1} \frac{\delta_{\alpha\rho}}{k^2} \frac{\delta_{\delta\beta}}{k'^2} \tag{10}$$

$$\frac{1}{(p^2+m^2)(p'^2+m^2)} \mathrm{Tr}\Big[\gamma_\rho(-i\gamma.p+m)\gamma_\delta(-i\gamma.p'+m)\Big] \, \delta(k-p+p')\,\delta(p-k'-p')\Bigg\}$$

3. ANALYTICAL CONTINUATION

First of all it is convenient to notice that the trace in ν dimensions indicated in expression (10) is given by

$$d(\nu)\,\left[-\,p_\rho p'_\delta - p'_\rho p_\delta + m^2\delta_{\rho\delta} + p.p\,\delta_{\rho\delta}\right] \tag{11}$$

where $d(\nu)$ is an analytic function of ν and, for ν integer, coincides with the number of components of the spinor in a ν-dimensional space[*].

It is convenient to introduce the Feynman parametrization

$$\frac{1}{(p^2+m^2)((p-k)^2+m^2)} = \int_0^1 \frac{dz}{(p_1^2+c^2)^2} \tag{12}$$

where

$$p_1 = p-k(1-z)$$

and

$$c^2 = k^2 z(1-z) + m^2$$

With the substitution of eq.(11) and (12) intoeq. (10), one obtains, after integrating in p', k', and p

$$\bar{W}_2 = \frac{e^2}{2} \oint dx_1^\alpha \oint dx_2^\beta \int \frac{d^\nu k}{(2\pi)^\nu} e^{ik(x_2-x_1)} \cdot \frac{1}{k^4} \cdot$$

$$\left\{ \frac{e^2\pi^{\nu/2} d(\nu)}{(2\pi)^\nu} \Gamma\left(\frac{4-\nu}{2}\right) \int_0^1 dz \, \frac{2(k_\alpha k_\beta + k^2\delta_{\alpha\beta}) z(1-z)}{|k^2 z(1-z) + m^2|^{(4-\nu)/2}} \right\} \tag{13}$$

The integration In p was carried out using the formulae of appendlx B of ref. 5 . Now the parametric integral can be performed wlth a convenient change of varlable and the use of the formulae (3.681-1,2) of ref. 7. We obtaln

$$\bar{W}_2 = \frac{e^2}{2} \oint dx_1^2 \oint dx_2^\beta \int \frac{d^\nu k}{(2\pi)^\nu} e^{ik(x_2-x_1)} \frac{1}{k^4} (k_\alpha k_\beta - k^2\delta_{\alpha\beta})$$

$$\left\{ \frac{e^2\pi^{\nu/2}}{(2\pi)^\nu} d(\nu) \, \Gamma\left(\frac{4-\nu}{2}\right) \frac{1}{3(m^2)^{(4-\nu)/2}} F\left[\frac{4-\nu}{2}, 2; \frac{5}{2}; -\frac{k^2}{4m^2}\right] \right\} \tag{14}$$

The hypergeometric function that appears in this expression is dlvergent for $k^2 = 4m^2$ and is equal to one for $\nu=4$. This divergence can be avolded if one uses the transformation formula (9.131-1) of ref. 7

$$F\left[\frac{4-\nu}{2}, 2; \frac{5}{2}; \frac{-k}{4m^2}\right] = \left[\frac{4m^2}{4m^2+k^2}\right]^{(4-\nu)/2} F\left[\frac{4-\nu}{2}, \frac{1}{2}; \frac{5}{2}; \frac{k^2}{k^2+4m^2}\right]$$

$$= \left[\frac{4m^2}{4m^2+k^2}\right]^{(4-\nu)/2} \left[1 + \sum_{n=1}^{\infty} \frac{\left(\frac{4-\nu}{2}\right)_n \left(\frac{1}{2}\right)_n (k^2)^n}{(5/2)_n \, n! \, (k^2+4m^2)^n}\right] \tag{15}$$

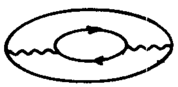
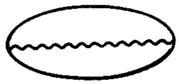
After the substitution of eq. (15) into eq. (14), one obtains the exact expresslon for the correction term $\bar{W}_2$ in eq. (10)

$$\bar{W}_2 = \frac{e^2}{2} \oint dx_1^\alpha \oint dx_2^\beta \int \frac{d^\nu k}{(2\pi)^\nu} e^{ik(x_2-x_1)} \frac{1}{k^4} (k_\alpha k_\beta - k^2\delta_{\alpha\beta}) \Gamma\left(\frac{4-\nu}{2}\right)$$

$$\left[\frac{e^2\pi^{\nu/2} d(\nu)}{3(2\pi)^2 (m^2)^{(4-\nu)/2}}\right] \left[\frac{4m^2}{4m^2+k^2}\right]^{\left(\frac{4-\nu}{2}\right)} \left[1 + \sum_{n=1}^{\infty} \frac{\left(\frac{4-\nu}{2}\right)_n \left(\frac{1}{2}\right)_n (k^2)^n}{\left(\frac{5}{2}\right)_n \, n! \, (k^2+4m^2)^n}\right] \tag{16}$$

4. VACUUM PQLARIZATION

Adding up eqs. (8) and (16), one has

 + 

$$= \frac{e^2}{2} \oint dx_1^\alpha \oint dx_2^\beta \left\{ \int \frac{d^\nu k}{(2\pi)^\nu}\, e^{ik(x_2-x_1)} \right.$$

$$\left[-\frac{\delta_{\alpha\beta}}{k^2} + \frac{1}{k^4}\left(k_\alpha k_\beta - k^2 \delta_{\alpha\beta}\right)\right] \left[\frac{e^2 \pi^{\nu/2} d(\nu)}{3(2\pi)^\nu (m^2)^{(4-\nu)/2}}\right] \qquad (17)$$

$$\left. \left(\frac{4m}{4m^2+k^2}\right)^{(4-\nu)/2} \Gamma\left(\frac{4-\nu}{2}\right) \left[1 + \sum_{n=1}^{\infty} \frac{\left(\frac{4-\nu}{2}\right)_n \left(\frac{1}{2}\right)_n (k^2)^n}{\left(\frac{5}{2}\right)_n\, n!\, (k^2+4m^2)^n}\right] \right\}$$

Then one obtains that

$$\bar{D}_{\alpha\beta}(x_1-x_2) = \int \frac{d^\nu k}{(2\pi)}\, e^{ik(x_2-x_1)} \left\{ -\frac{\delta_{\alpha\beta}}{k^2} + \frac{1}{k^4}\left(k_\alpha k_\beta - k^2 \delta_{\alpha\beta}\right) \right.$$

$$\frac{e^2\, d(\nu)}{3.2^\nu \pi^{\nu/2} (m^2)^{(4-\nu)/2}} \left(\frac{4m^2}{4m^2+k^2}\right)^{(4-\nu)/2} \qquad (18)$$

$$\left. \left[\Gamma\left(\frac{4-\nu}{2}\right) + \sum_{n=1}^{\infty} \Gamma\left(\frac{4-\nu}{2}\right) \frac{\left(\frac{4-\nu}{2}\right)_n \left(\frac{1}{2}\right)_n (k^2)^n}{\left(\frac{5}{2}\right)_n\, n!\, (k^2+4m^2)^n}\right] \right\}$$

is the new photon propagator with a correction due to the Insertlon of the second-order photon self-energy. The Fourier transforrn is now well deflned, because of the transforrnation (15). Hence one can write eq. (18) in momentum space as

$$\bar{D}_{\alpha\beta}(k) = D_{\alpha\beta}(k) \left\{ 1 - \frac{1}{k^2 \delta_{\alpha\beta}} \left(k^2 \delta_{\alpha\beta} - k_\alpha k_\beta\right) \left(\frac{e^2\, d(\nu)}{3.2^\nu \pi^{\nu/2} (m^2)^{(4-\nu)/2}}\right) \right.$$

$$\left. \left(\frac{4m^2}{4m^2+k^2}\right)^{(4-\nu)/2} \left[\Gamma\left(\frac{4-\nu}{2}\right) + \sum_{n=1}^{\infty} \left(\frac{4-\nu}{2}\right) \frac{\left(\frac{4-\nu}{2}\right)_n \left(\frac{1}{2}\right)_n (k^2)^n}{\left(\frac{5}{2}\right)_n\, n!\, (k^2+4m^2)^n}\right] \right\}$$

(19)

where

$$D_{\alpha\beta}(k) = -\frac{\delta_{\alpha\beta}}{k^2}$$

is the photon Green's function in the zero-order approximation. From eq. (19) one sees that the vacuum polarization in ν dimensions is given by

$$\pi_{\alpha\beta}(k,\nu) = \frac{e^2}{(4\pi)^{\nu/2}}\ \frac{d(\nu)}{3}\ \Gamma(2-\frac{\nu}{2})\ m^{(\nu-4)}\left[\frac{4m^2}{4m^2+k^2}\right]^{(4-\nu)/2}$$

$$(k_\alpha k_\beta - k^2\delta_{\alpha\beta})\left[1+\sum_{n=1}^{\infty}\frac{(\frac{4-\nu}{2})_n(\frac{1}{2})_n(k^2)^n}{(\frac{5}{2})_n\ n!\,(k^2+4m^2)}\right] \tag{20}$$

If one expands the series of the last equation around $\nu=4$, one obtains

$$\pi_{\alpha\beta}(k,\nu) = \frac{e^2}{(4\pi)^{\nu/2}}\ \frac{d(\nu)}{3}\ m^{(\nu-4)}\left[\frac{4m^2}{4m^2+k^2}\right]^{(4-\nu)/2}$$

$$\left[\Gamma(\frac{4-\nu}{2}) + \sum_{n=1}^{\infty}\Gamma(3-\frac{\nu}{2})\ \frac{(\frac{1}{2})_n}{(\frac{5}{2})_n\, n}\ \frac{(k^2)^n}{(k^2+4m^2)^n} + O(\frac{4-\nu}{2})\right](k_\alpha k_\beta - k^2\delta_{\alpha\beta}) \tag{21}$$

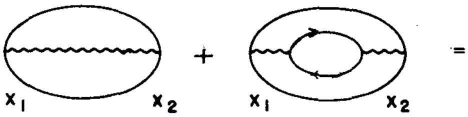
Notice now that the part of eq. (21) without pole is well defined for any momentum trapsfer in the limit of $\nu=4$.

5. THE RENORMALIZED CHARGE

From eq. (19) one obtains, for $k^2 \simeq 0$, an approximate relation between $\bar{D}_{\alpha\beta}(k)$ and $D_{\alpha\beta}(k)$

$$\bar{D}_{\alpha\beta}(k) = D_{\alpha\beta}(k)\left[1-\frac{e^2 d(\nu)}{3(4\pi)^{\nu/2}}\ m^{(\nu-4)/2}\ \Gamma(\frac{4-\nu}{2})\right] \tag{22}$$

Now, one can write



$$= -\frac{e^2}{2}\oint dx_1^{\alpha}\oint dx_2^{\beta}\int\frac{d^{\nu}k}{(2\pi)^{\nu}}\, e^{ik(x_2-x_1)}$$

$$D_{\alpha\beta}(k)\left[1-\frac{e^2 d(\nu)}{3(4\pi)^{\nu/2}}\, m^{(\nu-4)/2}\,\Gamma\left(\frac{4-\nu}{2}\right)\right] \tag{23}$$

Instead of regarding

$$\left[1-\frac{e^2 d(\nu)}{3(4\pi)^{\nu/2}}\, m^{(\nu-4)/2}\,\Gamma\left(\frac{4-\nu}{2}\right)\right] \tag{24}$$

as a correction factor to the photon propagator, one can alternatively regard it as a correction factor to the coupling constant. Notice that expression (23) is the same as the lowest-order expression (8) except that the electric charge is decreased as follows

$$e^2 \rightarrow e^2\left[1-\frac{e^2 d(\nu)}{3(4\pi)^{\nu/2}}\, m^{(\nu-4)/2}\,\Gamma\left(\frac{4-\nu}{2}\right)\right] \tag{25}$$

This is the observable physical charge[2,8] which is oftencalled the renormalized charge denoted by e_r. Since the calculation was carried out in ν dimensions, one replaces the dimensionful coupling constant e by a dimensionless one,

$$e \rightarrow e\mu^{(4-\nu)}$$

where μ is the traditional mass parameter of dimensional regularization.

If one takes the residue of eq. (19) in the limit of $\nu \rightarrow 4$, one has

$$\mathrm{Re}\,\bar{W}_2 = \frac{e^2}{2}\oint dx_1^{\alpha}\oint dx_2^{\beta}\int\frac{d^{\nu}k}{(2\pi)^4}\, e^{ik(x_2-x_1)}\,\frac{1}{k^4}\,(k_\alpha k_\beta - k^2\delta_{\alpha\beta})\,\frac{e^2}{12\pi^2} \tag{26}$$

With this expression, the approximate formula (22) can be written as

$$\bar{D}_{\alpha\beta}(k) = D_{\alpha\beta}(k)\left[1-\frac{e^2}{12\pi^2}\,\Gamma\left(\frac{4-\nu}{2}\right)\right] \tag{27}$$

and the renormalized charge is now given by

$$e_r^2 = e^2\left[1 - \frac{e^2}{12\pi^2}\,\Gamma\left(\frac{4-\nu}{2}\right)\right] \tag{28}$$

Notice that this expression has a pole in the limit $\nu \to 0$ instead of the well-know cut-off dependence obtained via Pauli-Villars larization[2,8]

$$e_r^2 = e^2\left[1 - \frac{e^2}{12\pi^2}\,\log\left(\frac{M^2}{m^2}\right)\right] \tag{29}$$

which is valid for $M^2 >> m^2$.

6. CONCLUSION

In this work the photon propagator, in ν dimensions, with correction due to its self-energy, is derived up to second order in momeota space without encountering any difficulties with the Fourier transform. From the corrected propagator, in momentum space, one can write, also in ν dimensions, an expression for the vacuum polarization tensor. Notice that the part of eq. (21) without pole is well defined for any momentum transfer and this result coincides, for small k^2, with the usual one.

The renormalized electric charge in ν dimensions is also obtained using the Wilson loop as a trick.

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Resumo

A expressão da polarização do vácuo é obtida para qualquer momentum transferido em ν dimensões. Usando o loop de Wilson para QED, a carga renormalizada em ν dimensões, é calculada.