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Dissipative Systems: Radiation Loss in the Quantized Field

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Abstract A simple model of a cavity coupled to the outside world and the use of appropriate boundary conditions provide the inclusion of dissipation in the quantized radiation field, in a non-artificial way. By defining collective operators which enable us to project the field intothe internal optical cavity we obtain the exponential decay for the field inside the cavity. The initial conditions that lead to the exponential damping and the lower and upper bounds to this exponential evolution are also investigated.

1. INTRODUCTION

Much effort has been done to develop a quantum theory of dissipative systems appearing in various branches of physics as atomic and nuclear physics[1], solid state physics[2], laser theory[3,4], general relativity[5], etc. A review article by Dekker[6] covers much of what has been done in this subject to date.

In Newtonian mechanics, the dissipation yields no conceptual problem: one inserts a friction term $f(v)$ in the equation of motion, where $f(v)$ is a function depending on the speed of the particle. In classical electrodynamics, an analogous procedure is applied through a phenomenological term σE introduced in Maxwell's equations, where σ is the ohmic conductivity of the (often fictitious) medium and E is the electric field.

The same does not occur in the classical framework of a Lagrangian or Hamiltonian formalism, where some difficulties emerge when dissipation is present. Also, these difficulties seemingly increase when canonical quantization is applied.

In quantum theory, the inclusion of dissipation has followed two main lines: (i) one starts from a classical equation of motion for a system with dissipation, finds the Lagrangean which leads to the equation and the then applles the canonical quantization method; (ii) one considers the dissipation as being due to the coupling between two systems: a dissipative system S and an absorptive system R, the so-called loss reservoir. The quantization is applied to the whole system $S + R$ which is conservative and the dissipative system (now a subsystem) is obtained by eliminating the variables of the system R.

In the first line (i), we refer to the approach of Bateman, Caldirola and Kanai[7-9]. In the second line (ii), there are several approaches including losses in quantum theory[10-15]: one couples the system of interest to another physical system so that the exchange of energy between them is properly described. In most pratical situations it is necessary to use a model for the loss mechanism since the actual mechanism is either too complicated to describe or even incompletely known. This difficulty in treating the actual mechanism leads one to assume artificial loss mechanisms. For later reference and comparison, we shortly give a formal presentation of this method, as follows: taking the Hamiltonian for the whole system $S + R$, as the sum

$$H = H_S + H_R + V_{SR} \tag{1.1}$$

where $H_S(H_R)$ is the free Hamiltonian for the system $S(R)$ and V_{SR} is the interaction between S and R, the Heisenberg equation for an operator $\hat{a} \in S$ is

$$\frac{d\hat{a}}{dt} = i\hbar[H,\hat{a}] = i\hbar[H_S,\hat{a}] + i\hbar[V_{SR},\hat{a}] \tag{1.2}$$

and, similarly, for an operator $\hat{b} \in R$

$$\frac{d\hat{b}}{dt} = i\hbar[H_R,\hat{b}] + i\hbar[V_{SR},\hat{b}] \tag{1.3}$$

By eliminating the operator $\hat{b}$ in the system of coupled equations (1.2), (1.3) one solves the system for the operator $\hat{a} \in S$ and obtains the irreversibility if the system R and the interaction V_{SR} have suitable properties with respect to the (often artificial) loss mechanism. One may argue, on this later point, for the possibility and advantage in treating a model which is able to include the loss mechanism in a neither phenomenological nor artificial way.

In this paper we show, by means of a simple model, how losses can be incorporated in a natural way and develop a quantum theory of damping where the physical system is the radiation field in an optical cavity. We point out some peculiarities in this procedure, where a realistic inclusion of the loss mechanism is exhibited and also compare some points in this treatment with those appearing in the line (ii), as mentioned in this section. In section 2, we describe the cavity model and obtain the normal modes of the optical cavity and, also, we quantize the radiation field in the entire cavity. In section 3 we define collective operators which allow us to project the field onto the subspace of the internal cavity, as defined in section 2. In section 4, we show how to obtain the damping in this quantized radiation field, and two physical situations are investigated: the field in pure and in mixed states. Finally, in section 5 we investigate the limitations, in time domain, to the exponential damping, as obtained in sec. 4.

2. MODEL, FIELD MODES AND QUANTIZATION

The optical cavity in this model, a somewhat modified version of similar models encountered in quantum[16] and semiclassical[17] laser theory, is a free space region bounded by two parallel plates, one of which is totally reflecting, whereas the other one is semitransparent. We take the origin at the semitransparent plate and the x-axis perpendicular to the plates. Taking R as the separation between the plates, the optical cavity is the region $0 \leqslant x \leqslant R$, and the outside region is the left half-space: $-\infty \leqslant x \leqslant 0$.

In order to avoid ohmic losses, we take the plate coating of the semitransparent window as a dielectric film rather than a half-silvered window. The film is assumed in the model as a limiting case of a very thin layer with a large dielectric constant, described by[17c]

$$\varepsilon(x) = \varepsilon_0 \left[1 + \eta\delta(x)\right] \qquad (2.1)$$

where ε_0 is the electrical permissivity of vacuum; η is a real parameter with dimension of length, which determines the transparency of the window at $x = 0$, and $\delta(x)$ is the Dirac δ-function.

In this model the dissipation is entirely due to the transmission through the window from the inside region to the outside one, which plays the role of a loss reservoir. By considering only longitudinal field modes, the model becomes one-dimensional in the sense that the field depends only on the x-coordinate.

The normal field modes are stationary solutions of Maxwell's equations with appropriate boundary conditions and constituteacontinuous spectruni. They are given by[17c]

$$U_k(x) = U_k^{(in)}(x) = L_k \sin[k(x - \ell)], \quad x \in [0,\ell]$$

and

$$U_k(x) = U_k^{(out)}(x) = \sqrt{2/\pi}\,\sin(kx - \phi_k), \quad x \in (-\infty,0] \tag{2.2}$$

where we have employed the 6-function normalization

$$\int_{-\infty}^{\ell} [\varepsilon(x)/\varepsilon_0]\; U_k(x)\; U_{k'}(x)dx = \delta(k - k') \tag{2.3}$$

and ϕ_k is the phase-shift and L_k is given below in eq. (2.4). The plot of L_k^2 as a function of frequency $\omega_k(=ck)$, shows that the resonances are centered approximately around the Fox-Li quasi-mode frequencies, with spacing $\Delta\omega = c\pi/\ell$.

If the transrnlssion through the window is very small then the function L_k^2 will be strongly peaked around the Fox-Li quasi-mode frequencies. In this case, the line width Γ_n associated with a given Fox--Li resonance frequency ω_{0n}, is much smaller than the spacing between neighboring resonances, i.e., $\Gamma_n \ll \Delta\omega = c\pi/\ell$ and we can approximate the line-shape function L_k by a Lorentzian function $M_k(n)$ as[17c]

$$L_k \simeq M_k(n) = (2/\pi)^{1/2}\; \Gamma_n \Lambda_{0n} / [(\omega_k - \omega_{0n})^2 + \Gamma_n^2]^{1/2} \tag{2.4}$$

where Γ_n is determined by the window transparency[17c]

$$\Gamma_n = c/(\Lambda_{0n}^2 \ell) \tag{2.5}$$

with

$$\Lambda_{0n} = n(\eta\pi/\ell) \gg 1 \tag{2.6}$$

and ω_{0n} is the resonance frequency associated with the Fox-Li quasi-mode (in the optical case $n \simeq 10^6 \gg 1$) given by

$$\omega_{0n} = ck_{0n} \simeq (n\pi + \Lambda_{0n}^{-1})c/\ell \tag{2.7}$$

Eq.(2.6) expresses the requirement that the transparency of the window is very small. The reader should turn to ref. 17c for further details.

By using the normal field modes (eq. (2.2)) and following the usual quantlzation procedure[18], we find the Hamiltonian for the radiation field in the'entire cavity ($x \in (-\infty;\ell]$)

$$H = \int_0^\infty \omega_k a_k^+ a_k dk \; ; \qquad \hbar = 1 \tag{2.8}$$

where the zero point energy has been neglected and $a_k^+ (a_k)$ creates (annihilates) photons with momentum k, in the whole space $x \in (-\infty, \ell]$,with the standard canonical commutation relations

$$\begin{aligned} [a_k, a_{k'}] &= [a_k^+, a_{k'}^+] = 0 \\ [a_k, a_{k'}^+] &= \delta(k-k') \end{aligned} \tag{2.9}$$

The Hamiltonian (2.8) does not depend on two types of operators:$a^+(a) \in S$ and $b^+(b) \in R$, as in the artificial procedure where $H = H_S + H_R + V_{SR}$ (cf. eq. (1.1)), and this forbides one to handle the usual technique, as refered to in sec 1, which essentially comes from the elimination of the reservoir varlables b_k^+, b_k, to get the open system S.

Hence, a question which emerges is how could one eliminate in eq. (2.8) the *outside* region to obtain the open, irreversible, system. In order to solve this question, we define collectlve operators in the following section.

3. COLLECTIVE OPERATORS FOR THE INTERNAL FIELD

The electric field operator, expressed in terms of the operatorr a_k, a_k^+ and the normal field modes, is the superposition

$$E(x,t) = \int_0^\infty E_{0k}(a_k^+ + a_k) U_k(x) dk \tag{3.1}$$

where $E_{0k} = [\omega_k/(2\varepsilon_0)]^{1/2}$, $\omega_k = ck$. In order to define collective operators for the field inslde the cavity we refer to the electric field $E^{(i)}(x,t)$, $x \in [0,\ell]$ as

$$\begin{aligned} E^{(i)}(x,t) &= \int_0^\infty E_{0k}(a_k^+ + a_k) U_k^{(i)}(x) dk \\ &= \int_0^\infty E_{0k}(a_k^+ + a_k) L_k \sin[k(x-\ell)] dk \end{aligned} \tag{3.2}$$

Next, we make cuts in the iineshape function L_k and divide it in bands B_n having linewidth $\Delta\omega = c\pi/\ell$, each band centered in the Fox-Li quasi--modes ω_{0n}. In this way, as mentioned before, we replace $L_k \simeq M_k(n)$, in the neighborhood of ω_{0n} and set

$$M'_k(n) = M_k(n)\,\chi_k(n) \tag{3.3}$$

where $\chi_k(n)$ is the characteristic function of the interval

$$\chi_k(n) = \begin{matrix} 1 \ , & k \in B_n \\ 0 \ , & k \notin B_n \end{matrix}$$

With this choice in hand we have

$$M'_k(n)M'_k(n') = 0 \ , \quad n' \neq n$$

and eq. (3.2) can be written in the form

$$\begin{aligned} E^{(in)}(x,t) &= \sum_n \int_{B_n} E_{0k}(a_k^+ + a_k)M'_k(n) \sin\left[k(x-\ell)\right] \\ &\simeq \sum_n E_{0k_n} \sin\left[k_n(x-\ell)\right] \left\{\int M'_k(n)\,(a_k^+ + a_k)\,dk\right\} \end{aligned} \tag{3.4}$$

where the approximatlon

$$E_{0k} \sin\left[k(x-\ell)\right] M'_k(n) \simeq E_{0k_n} \sin\left[k_n(x-\ell)\right] M'_k(n)$$

has been employed, since $E_{0k} \sin\left[k(x-\ell)\right]$ is a slowly varying function when compared with the strongly peaked Lorentzian function $M'_k(n)$. So, instead of performlng the integral as in eq. (3.2) we integrate in a generic band B_n and afterwards sum over all bands. At this point we define the collective operators

$$A_n = (1/M) \int_{B_n} M'_k(n)\,a_k\,dk \tag{3.5}$$

$$A_n^+ = (1/M) \int_{B_n} M'_k(n)\,a_k^+\,dk \tag{3.6}$$

where M is a normalization factor and A_n^+ (A_n) creates (annihilates) photons in the band B_n. Immediate application of eqs. (3.5) and (3.6) in eq. (3.4) gives the internal field in terms of these collective operators

$$E^{(in)}(x,t) = \sum_n ME_{0k_n} \sin[\bar{k}_n(x-\ell)] (A_n^+ + A_n) \tag{3.7}$$

By using the relations (2.9) and setting the normalization factor

$$M = \left(\int_{B_n} M_k^2(n) dk \right)^{1/2}$$

we readily find

$$\begin{aligned} [A_n, A_n] &= [A_n^+, A_n^+] = 0 \\ [A_n, A_{n'}^+] &= \delta_{n,n'} \end{aligned} \tag{3.8}$$

According to eq. (3.7), the internal field depends only on the collective operators. In the following section we show that they have the dissipation already built-in.

4. DISSIPATION IN THE INTERNAL FIELD

a) Field in Pure State

Let us assume that the field is in a pure state $|\Psi_{0n}>$ of only a single collective mode and let us take the Heisenberg representation to calculate the expectation value of the collective operator

$$\begin{aligned} <A_n>(t) &= <\Psi_{0n}|A_n(t)|\Psi_{0n}> \\ &= \int_{B_n} M_k(n) e^{-i\omega_k t} <\Psi_{0n}|a_k|\Psi_{0n}> dk \end{aligned} \tag{4.1}$$

where eq. (3.5) has been used. If we also assume the pure state satisfying

$$<\Psi_{0n}|a_k|\Psi_{0n}> = c_0 M_k(n) \tag{4.2}$$

where c_0 is an arbitrary constant, we have

$$<A_n>(t) = c_0 \int_{B_n} M_k^2(n) e^{-i\omega_k t} dk \tag{4.3}$$

Substituting eq. (2.4) into eq. (4.3) and extending the domain $k \in B_n$ to $k \in (-\infty,+\infty)$, which is reasonable for $\Gamma_n \ll \Delta\omega$ and ω_{0n} within the optical domain ($\omega_{0n} \simeq 10^{15}$ Hz), we find

$$\langle A_n(t)\rangle = (2c_0/\ell)e^{-i\omega_{0n}t}e^{-\Gamma_n t} \tag{4.4}$$

In a similar way, we find

$$\langle A_n^+\rangle(t) = (2c_0^*/\ell)e^{i\omega_{0n}t - \Gamma_n t} \tag{4.5}$$

Substitutiny eqs. (4.4) and (4.5) back into eq. (3.7) we obtain

$$E^{(in)}(x,t) = \{c_0 \sin[k_n(x-\ell)]e^{i\omega_{0n}t} + \text{c.c.}\}\, e^{-\Gamma_n t} \tag{4.6}$$

whicn shows an exponentlal damping in the expectation value of the field inside the cavity. Hence, the dissipation was included in the field without any phenomenological or artificial procedure.

One may ask about the nature and the possibility of having a pure state corresponding to the condition (4.2) leading to the exponential evolutlon (cf. (4.6)). In order to investigate this point, let us assume the field in a coherent state in the whole cavlty, so that

$$|\Psi_{0n}\rangle = \{|v_k\rangle\}$$

$$a_k\{|v_k\rangle\} = v_k\{|v_k\rangle\} \tag{4.7}$$

yielding

$$\langle\Psi_{0n}|a_k|\Psi_{0n}\rangle = v_k \tag{4.8}$$

and the conditlon (4.2) is accomplished with the choice $v_k = c, M_k(n)$. Hence, the Initial condition (4.2) may correspond to a field in a coherent state with an elgenvalue distribution $\{v_k\}$ following the Lorentzian lineshape $M_k(n)$, defined by the optical cavity in the present model. Clearly, neither a field in Fock-state nor a thermal field can satisfy eq. (4.2).

b) Field in Mixed State

In this section, in order to avoid confusion with the number-representation, we replace the notation $A_n \to A$. The density operator $\rho^{(in)}$, describing the internal field in a mixed state, depends only on the collective operators A^+ and A. Its most general form is

$$\rho^{(in)} = \sum_{rs} \sum_{\{\alpha_i\}} \sum_{\{\alpha_j'\}} c_{rs}(\alpha_1,\ldots,\alpha_r,\alpha_1',\ldots,\alpha_s') \prod_{\alpha_i=1}^{r} A_{\alpha_i}^+ |0\rangle\langle 0| \prod_{\alpha_j'=1}^{s} A_{\alpha_j'} \tag{4.9}$$

The operator

$$\xi_{\alpha}^{(m,n)}(t) = (A_{\alpha}^{+})^{m}(t)A_{\alpha}^{n}(t) \tag{4.10}$$

allows us to evaluate the coherence functions of all orders for the internal field in a single eollective mode a, namely

$$G_{\alpha}^{(m,n)}(t) = \mathrm{tr}\{\xi_{\alpha}^{(m,n)}(t)\rho^{(in)}(0)\}$$
$$= \mathrm{tr}\{\xi_{\alpha}^{(m,n)}(0)\rho^{(in)}(t)\} \tag{4.11}$$

where in the first equality we use the Heisenberg representation and in the second we use the Schrödinger one.

In order to obtain the equation of motion for the operator $\rho^{(in)}(t)$ we first evaluate the coherence functions $G_{\alpha}^{(m,n)}(t)$ starting from the initial state $\rho^{(in)}(0)$, as given in eq. (4.9). To this end we will use the first form In eq. (4.11) which depends on $\rho^{(in)}(0)$. The second form in eq. (4.1 1) allows one to find $\rho^{(in)}(t)$ corresponding to the same coherence function $G_{\alpha}^{(m,n)}(t)$. According to rhe reconstrustion theorem for the electromagnetic field[19], such operator $\rho^{(in)}(t)$ satisfying eq. (4.11) is the density operator for the radiation field inside the cavity.

For algebraic reasons, we write (4.10) in the more general form

$$\xi_{\beta\gamma}^{(m,n)}(t) = N\{\prod_{\beta_i\gamma_j}^{m,n} A_{\beta_i}^{+}(t)A_{\gamma_j}(t)\} \tag{4.12}$$

where N is the normal ordering operator[20], and find

$$G_{\beta\gamma}^{(m,n)}(t) = \langle\xi_{\beta\gamma}^{(m,n)}(t)\rangle = \mathrm{tr}\{\rho^{(in)}(0)\xi_{\beta\gamma}^{(m,n)}(t)\} =$$

$$= \mathrm{tr}\{\sum_{rs}\sum_{\{\alpha_i\}}\sum_{\{\alpha'_j\}} c_{rs}(\alpha_1,\dots,\alpha_r;\alpha'_1,\dots,\alpha'_s)\,\xi_{\beta\gamma}^{(m,n)}(t)A_{\alpha_1}^{+}\dots A_{\alpha_r}^{+}|0\rangle\langle 0|A_{\alpha_1}\dots A_{\alpha'_s}\}$$

which can be written in the form

$$\langle\xi_{\beta\gamma}^{(m,n)}(t)\rangle = \sum_{rs}\sum_{\{\alpha_i\}}\sum_{\{\alpha'_j\}} c_{rs}(\alpha_1,\dots,\alpha_r;\alpha'_1,\dots,\alpha'_s)$$

$$\cdot\ \mathrm{tr}\ (A_{\alpha'_1},\dots,A_{\alpha'_s})(A_{\beta_1}^{+}\dots A_{\beta_m}^{+})(A_{\gamma_1}\dots A_{\gamma_n})(A_{\alpha_1}^{+}\dots A_{\alpha_r}^{+})\}|0\rangle\langle 0|$$

$$= \sum_{rs}\sum_{\{\alpha_i\}}\sum_{\{\alpha'_j\}} c_{rs}(\alpha_1\dots\alpha'_s)\ \langle 0|\hat{\mathbf{0}}|0\rangle \tag{4.13}$$

where $\hat{0}$ is the operator

$$\hat{0} = (A_{\alpha_1}(0)\ldots A_{\alpha_s'}(0))(A^+_{\beta_1}(t)\ldots A^+_{\beta_m}(t))(A_{\gamma_1}(t)\ldots A_{\gamma_n}(t))(A^+_{\alpha_1}(0)\ldots A^+_{\alpha_r}(0)) \quad (4.14)$$

The evaluation of $\langle 0|\hat{0}|0\rangle$ in eq. (4.13) is performed through the use of the Wick's theorem[20]. From eqs. (3.5) and (3.6) we have

$$[A_\alpha(t), A_{\alpha'}(t')] = [A^+_\alpha(t), A^+_{\alpha'}(t')] = 0 \quad (4.15)$$

and

$$[A_\alpha(t), A^+_{\alpha'}(t')] = e^{i\omega_\alpha(t-t')}\, e^{-\Gamma_\alpha|t-t'|}\,\delta_{\alpha,\alpha'} \quad (4.16)$$

The application of eqs. (4.15) and (4.16), appearing in the Wick's theorem, to eq. (4.13) allows us to find, after a lengthy calculation,

$$\langle\xi^{(m,n)}_{\beta\gamma}(t)\rangle = \{e^{i\omega_{\beta_1}t-\Gamma_{\beta_1}t}\, e^{i\omega_{\beta_2}t-\Gamma_{\beta_2}t}\ldots e^{i\omega_{\beta_n}t-\Gamma_{\beta_n}t}\}$$

$$\cdot\, e^{-i\omega_{\gamma_1}t-\Gamma_{\gamma_1}t}\, e^{-i\omega_{\gamma_2}t-\Gamma_{\gamma_2}t}\ldots e^{-i\omega_{\gamma_n}t-\Gamma_{\gamma_n}t}\,\langle\xi^{(m,n)}_{\beta\gamma}(0)\rangle \quad (4.17)$$

At this point we specialize eq. (4.12) to the case of eq. (4.10) and apply eq. (4.17) to obtain

$$\langle\xi^{(m,n)}_\alpha\rangle(t) = e^{i(m-n)\omega_\alpha t}\, e^{-(m+n)\Gamma_\alpha t}\,\langle\xi^{(m,n)}_\alpha\rangle(0) \quad (4.18)$$

Next, we use again eq. (4.11), in Schrödinger representation, to find the operator $\rho^{(in)}(t)$ which generates the same coherence functions as in eq. (4.18). One verifies that eq. (4.17) gives the time evolution prescribed by the following equation for the density operator

$$\dot{\rho}^{(in)}(t) = iL\rho^{(in)}(t) + \Lambda\rho^{(in)}(t) \quad , \quad \hbar = 1 \quad (4.19)$$

where L is the Liouville operator

$$L\rho^{(in)}(t) = [H, \rho^{(in)}(t)] \quad (4.20)$$

where H is the effective Hamiltonian for the internal field

$$H = \sum_a \omega_\alpha A^+_\alpha A_\alpha \quad (4.21)$$

where A = 1, and Λ is the loss operator

$$\Lambda\rho^{(in)} = \sum_\alpha \Gamma_\alpha [A_\alpha, \rho^{(in)} A_\alpha^+] + \text{h.c.}, \tag{4.22}$$

where Γ_α is given as in eq. (2.5). The solution of the eq. (4.19) is readily found as

$$\rho^{(in)}(t) = e^{-iLt}\, e^{\Lambda t}\, \rho^{(in)}(0) \tag{4.23}$$

where the property $[L,\Lambda] = 0$ has been used. Setting

$$\exp(\Lambda t)\ \rho^{(in)}(0) = V$$

eq. (4.23) becomes

$$\rho^{(in)}(t) = e^{-iLt}\, W \tag{4.24}$$

From eqs. (4.11) and (4.24) we obtain

$$<(A_\alpha^+)^m A_\alpha^n>(t) = \mathrm{tr}\{a^{(m,n)}(0)\ e^{-iLt}\ W\} \tag{4.25}$$

and using the identities

$$\mathrm{tr}\{(A_\alpha^+)^m\ A_\alpha^n\ e^{-iLt}\ W\} = e^{-i(n-m)\omega_\alpha t}\,\mathrm{tr}\{(A_\alpha^+)^m\ A_\alpha^n\ W\} \tag{4.26}$$

and

$$\mathrm{tr}\,\{(A_\alpha^\pm)^m\ A_\alpha^n \Lambda^j\ \rho^{(in)}(0)\} = [-\Gamma_\alpha(m+n)]^j\ \mathrm{tr}\{(A_\alpha^+)^m\ A_\alpha^n\ \rho^{(in)}(0)\} \tag{4.27}$$

we find

$$<(A_\alpha^+)^m\ A_\alpha^n>(t) = e^{i(m-n)\omega_\alpha t}\, e^{-(n+m)\Gamma_\alpha t}\ <(A_\alpha^+)^m A_\alpha^n>(0) \tag{4.28}$$

which coincides with eq. (4.18). According to the reconstruction theorem, the operator $\rho^{(in)}(t)$, given in eq. (4.23), is the density operator for the internal field. Also, according to eq. (4.23), (4.18) and (4.27), the dissipation is included in the internal field in a mixed state formalism and the damping follows an exponential behaviour.

The above procedure, through the use of Wick's theorem plus eq. (4.10) leading to eq. (4.28), corresponds to specify the initial state, eq. (4.9), to the case of a single collective mode, namely

$$\rho^{(in)}(0) = \sum_{rs} c_{rs}(\alpha)\,(A_\alpha^+)^r\ |0><0|A_\alpha^s$$
$$= \sum_{rs} c_{rs}(\alpha)\ \sqrt{r!}\,\sqrt{s!}\ |r><s| = \sum_{rs} \rho_{r,s}\ |r><s| \tag{4.29}$$

One may also ask for the nature of this initial state yielding the exponential decay. In order to give an answer to this point, let us assume the internal field in a collective coherent state

$$\rho^{(in)}(0) = |v_\alpha\rangle\langle v_\alpha| \tag{4.30}$$

where[21]

$$|v_\alpha\rangle = \left(e^{-|v_\alpha|^2/2}\right) \sum_r \frac{(v_\alpha)^r}{\sqrt{r!}} |r\rangle \tag{4.31}$$

Replacing eq. (4.31) into (4.30) we find

$$\rho^{(in)}(0) = \sum_{r,s} e^{-|v_\alpha|^2} \frac{v_\alpha^r (v_\alpha^*)^s}{\sqrt{r!}\sqrt{s!}} |r\rangle\langle s| \tag{4.32}$$

which coincides with eq. (4.29) for

$$c_{r,s} = \frac{1}{r!}\frac{1}{s!} e^{-|v_\alpha|^2} v_\alpha^r (v_\alpha^*)^s \tag{4.33}$$

The expectation value of the internal field is readity found as

$$\begin{aligned}\langle E^{(in)}(x,t)\rangle &= \mathrm{tr}(\rho^{(i)}(0)E^{(i)}(x,t)) \\ &= c_0 \sin[k_\alpha(x-\ell)] \sum_r \{\sqrt{r+1}\, \rho_{r,r+1}\, e^{i\omega_\alpha t} e^{-\Gamma_\alpha t} + \mathrm{c.c}\}\end{aligned} \tag{4.34}$$

where

$$\rho_{r,r+1} = e^{-|v_\alpha|^2} \frac{(v_\alpha)^r (v_\alpha^*)^{r+1}}{\sqrt{r!}\sqrt{(r+1)!}} \tag{4.35}$$

Replacing eq. (4.35) into eq. (4.34), we get

$$\langle E^{(in)}(x,t)\rangle = \{c_0' \sin[k_\alpha(x-\ell)]\, e^{i\omega_\alpha t} + \mathrm{c.c.}\}\, e^{-\Gamma_\alpha t} \tag{4.36}$$

where c_0' is a constant. This result (compare it with eq. (4.6)) shows that the Initial state in eq. (4.29) may be realized as a coherent state. Also, applying eq. (4.32) into eq. (4.11), it is easy to show that the coherence functions of all orders decay exponentially.

Next, let us assume that the field is in a thermal state. In this case, the density operator reads ($\hbar$=1)

$$\rho^{(in)}(0) = (1-e^{-\beta\omega_\alpha}) \sum_{n_\alpha=0}^{\infty} (e^{-\beta\omega_\alpha})^{n_\alpha} |n_\alpha\rangle\langle n_\alpha| \quad ; \quad \beta=1/kT \tag{4.37}$$

and

$$\rho_{n_\alpha,n'_\alpha} = (1-\xi_\alpha)\ \xi_\alpha\ \delta_{n_\alpha,n'_\alpha}\quad ;\ \xi_\alpha = e^{-\beta\omega_\alpha} \tag{4.38}$$

So, the density operator is diagonal in the Fock's basis and, according to eq. (4.34), we have

$$<F^{(in)}(x,t)> = 0$$

However, the coherence function for the field in a thermal state is readily found to be

$$G_\alpha^{(n,m)} = n_\alpha!\ (\bar{n}_\alpha)^{n_\alpha}\ \delta_{n_\alpha,m_\alpha}\ ;\quad \bar{n}_\alpha = kT/\omega_\alpha \tag{4.39}$$

and one could use it to verify the damping, say, in the field intensity, which is proportional[21] to $G_\alpha^{(1,1)}(t)$. In fact, immediate use of eqs. (4.11) and (4.28), for $m = n = 1$, leads to

$$G_\alpha^{(1,1)}(t) = <A_\alpha^+ A_\alpha>\ (t) = e^{-2\Gamma t} <A_\alpha^+ A_\alpha>(0) \tag{4.40}$$

This result shows that the initial state in eq. (4.29) may also be realized as field in thermal state.

5. LOWER AND UPPER BOUNDS FOR THE EXPONENTIAL DAMPING

In this sectlon we investigate the limitations, intime domain, to the validity of the exponential damping due to the present lossy-cavity. In order to find the lower bound we calculate the commutator

$$[A_n(0), A_{n'}^+(t)] = (\Gamma_n/\pi)\delta_{n,n'}\ e^{i\omega_{0n}t}\int_{B_n} M_k(n) e^{i(\omega_k-\omega_{0n})t}\, dk \tag{5.1}$$

where eqs. (2.9), (3.5), (3.6) and (3.9) have been used. Substituting eq. (2.4) in the foregoing equation we have

$$[A_n(0), A_{n'}^+(t)] = (\Gamma_n/\pi)\delta_{n,n'}\ e^{i\omega_{0n}t}.t.\int_{-\Delta\omega t/2}^{\Delta\omega t/2} [e^{iu}/(u^2+\Gamma_n^2 t^2)]\, du \tag{5.2}$$

In the limit $\Delta\omega t/2 \to \infty$, eq. (5.2) leads to the exponential behaviour

$$[A_n(0), A_{n'}^+(t)] = e^{i\omega_{0n}t}\ e^{-\Gamma_n t}\ \delta_{n,n'} \tag{5.3}$$

However, taking into account that[22]

$$\int_{-\xi}^{\xi} [e^{iu}/(u^2+\Gamma_n^2 t^2)]du = [i/(2\Gamma_n t)]e^{-\Gamma_n t} [E_1(-\Gamma_n t-iu) - E_1(\Gamma_n t-iu)] \quad (5.4)$$

where $E_1(z)$ is the exponential integral[22]

$$E_1(z) = \int_z^{\infty} (e^{-s}/s)\, ds \quad ; \quad \arg|z| < \pi \quad (5.5)$$

whose asymptotic expansion is

$$E_1(z) \simeq (e^{-z}/z)\left[1 - \frac{1}{z} + \frac{2!}{z^2} - \frac{3!}{z^3} + \ldots\right] \quad ; \quad |z| \gg 1 \quad (5.6)$$

we see that, for values of t outside the asymptotic domain, i.e., $\xi = \Delta\omega t/2 \lesssim 1$, the exponential evolution ceases to be valid. Setting $\Delta\omega\, t/2 \simeq 1$ we get $t \simeq \tau_0$, where $\tau_0 = 1/\Delta\omega \simeq \ell/c$ is roughly the time that light takes to travel the internal cavity. Hence, τ_0 is a characteristic time (lower bound) and the exponential damping in the internal field is valid for times t satisfying: $t \gg \tau_0$ (a typical value for τ_0 is 10^{-9} sec).

Next, we find the upper bound. To this end we rewrite eq. (5.1) in the form

$$[A_n(0), A_{n'}^+(t)] = \delta_{n,n'}(\Gamma_n/\pi)e^{i\omega_0 t} . t . \{\int_{-\infty}^{+\infty} [e^{iu}/(u^2+\Gamma_n t^2)]du - \int_{-\infty}^{-\Delta\omega t/2} [e^{iu}/(u^2+\Gamma_n^2 t^2)]du - \int_{\Delta\omega t/2}^{\infty} [e^{iu}/(u^2+\Gamma_n^2 t^2)]du\} \quad (5.7)$$

In the limit $\Delta\omega\, t/2 \to \infty$ we can neglect the second and third integrals in eq. (5.7), and the first integral recovers eq. (5.3). However, for $(\Delta\omega t/2) \gg 1$, we integrate eq. (5.7) by parts and find

$$\int_{\Delta\omega t/2}^{\infty} [e^{i}/(u^2+\Gamma_n^2 t^2)]du = (i/t)\, \{e^{i\Delta\omega t/2}/[(\Delta\omega/2)^2 + \Gamma_n^2][1 + O(t^{-1})]\}$$

$$\simeq (i/\Delta\omega)\, [e^{i\Delta\omega t/2}/(\Delta\omega t/2)]\, [1 + O(t^{-1})] \quad (5.8)$$

Substituting eq. (5.8) and its similar (the integral in the domain $(-\infty, -\Delta\omega t/2])$ in eq. (5.7), we find

$$[A_n(0), A^+_{n'}(t)] = e^{i\omega_{0n}t}\, e^{-\Gamma_n t} \{1 + \frac{4\Gamma_n}{\pi\Delta\omega}\, e^{\Gamma_n t} \frac{\sin(\Delta\omega t/2)}{(\Delta\omega t/2)} [1 + O(t^{-1})] \tag{5.9}$$

which shows the validity of the exponential damping in the internal field if

$$\frac{8}{\pi} \cdot \frac{\Gamma_n}{\Delta\omega} \frac{e^{\Gamma_n t}}{\Delta\omega\, t} \ll 1 \tag{5.10}$$

yielding

$$\Gamma_n t = (t/\tau_n) \ll 2 \log(\Delta\omega t/2) + \log[2 \log(\Delta\omega t/2)] \tag{5.11}$$

where $\tau_n = \Gamma_n^{-1}$ is the lifetime of the field in the internal cavity, with central angular frequency ω_{0n}. According to eqs. (2.5) and (2.7) one has

$$\tau_n = \Lambda_n^2(\ell/c) = \Lambda_n^2\, \tau_0 \tag{5.12}$$

On the other hand, $\Delta\omega = c\pi/\ell$ and the substitution of these results in eq. (5.11) allows one to estimate the upper bound τ_1 for the exponential damping. So, if we take into account the inequality (2.6) and set $\Lambda_n \simeq 10^2$, we find $\tau_1 \simeq 20\tau_n \simeq 10^5\, \tau_0$. Hence, combining the results of this section we get the time interval for the validity of the exponential behaviour as

$$10^2\tau_0 < t < 10^5\tau_0 \quad ; \quad \Lambda_n \gtrsim 10^2 \tag{5.13}$$

or, in terms of the lifetime τ_n

$$10^{-2}\, \tau_n < t < 20\, \tau_n \tag{5.14}$$

6. CONCLUDING REMARKS

The present model and procedure include the dissipation in a natural manner, in the quantized radiation field, through the application of appropriate boundary conditions on the field, with subsequent quantization plus the use of collective operators and the reconstruction theorem. The collective operators are suitably weighted and normalized superpositions of the usual creation and annihilation operators for modes of the continuous spectrum, extended over each single Fox-Li band. The projection onto the internal cavity is an indirect procedure similar to that employed by Bonifacio and Lugiato[23] in the theory of superradiance: one shows that the density operator arises entirely from the transmission losses; the application of the reconstruction theorem shows that this equation yields the correct set of coherence functions of all orders. This is only true, however, within the domain of validity of the exponential decay law for the temporal coherence functions.

The main results which emerge from this treatment are: (i) the replacement of individual operators, appearing in the artificial treatments (cf. e.g., eq. (1.2)), by collective operators, as in secs. 3, 4; (ii) the possibility of obtaining a description for the field inside and *also* outside the laser cavity (cf. eqs. (2.2) plus eq. (3.1)), not possible in the artificial treatments; (iii) the investigation on the initial conditions that leads to the exponential damping (cf. sec. 5.a and 5.b)); and (iv) the setting up of the limitations for this exponential evolution (cf. eq. (5.14)).

As a final rernark, we mention that the present description in terms of the collective operators A A^+ follows the line of ref. 17c and describes the field inside the cavity regardless of what is happening outside the cavity. However, an alternative definition of collective operators, following the line of ref. 16b, is able to separate clearly what is happening inside and outside the cavity, and a paper outlining this slightly modified description will be published elsewhere.

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Resumo

Um modelo simples de cavidade optica acoplada ao exterior e o uso de condições de contorno apropriadas permitem a inclusão da dissipação no campo de radiação quantizado, de maneira não artificial. Operadores coletivos convenientemente definidos levam-nos ao campo projetado na cavidade interna onde obtemos o decaimento exponencial do campo quantizado nessa região. As condições iniciais que levam a essa evolução exponencial, bem como suas limitações temporais Inferior e superior são também investigadas.