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Path Integration of the Forced Oscillator with a two-time Quadratic Action

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Abstract We first show that the path integration of the forced harmonic oscillator with a two-time quadratic action characterising memory effects can be given in terms of the solutions of certain integro-differential equations. The exact propagtor in closed form is then obtained for the specific kernel Introduced by Feynman in the polaron problem.

1. INTRODUCTION

In Feynman's formulation of nonrelativistic quantum mechanics, the propagator is expressed as

$$K(x,T;x_0,0) = \int_{x(0)=x_0}^{x(T)=x} \exp\{(i/\hbar)S[x(t)]\}D[x(t)] \tag{1}$$

where $D[x(t)]$ is designed to Indicate that the integral is over all paths with fixed end points (x,T) and $(x_0,0)$. Using the polygonal paths approach (Feynmann and Hibbs[1], 1965), the path integral in eq. (1) becomes

$$K(x,T;x_0,0) = \lim_{N\to\infty} (m/2\pi i\hbar\varepsilon)^{N/2} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \exp\{(i/\hbar)S_N\} \prod_{j=1}^{N-1} dx_j \tag{2}$$

where S_N is the discretised form of the action $S[x(t)]$ over the partition of the time interval into N subintervals of length ε. We have set $\varepsilon = T/N$ and $x_j = x(j\varepsilon)$. As is well known for quadratic action, the gaussian integral in eq. (2) can be transformed into the form

$$K(x,T;x_0,0) = \lim_{N\to\infty} [m/2\pi i\hbar\varepsilon \det(P)]^{1/2} \exp\{-\frac{im}{2\hbar\varepsilon}[(Y,U) - x_0^2 - x_N^2]\} \tag{3}$$

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with the help of the general formula

$$\int_{-\infty}^{\infty} \exp\{-[(X,PX) + a(X,Y)]\}dX = [\pi^{N-1}/\det(P)]^{1/2} \exp\{ \frac{1}{4} a^2(Y,U)\} \qquad (4)$$

Here P is an $(N-1)$-dimensional square matrix, Y is a column matrix having $(N-1)$ components and $U = P^{-1}Y$. Now the difficulties in evaluating eq. (3) are to calculate the exponent and the normalisation factor exactly in the limit as $\varepsilon \to 0$ or $N \to \infty$. However, eq.(3) has been evaluated exactly for the time-dependent harmonic oscillator (Montroll[2], 1952), for the time-dependent forced harmonic oscillator (Cheng[3], 1983) and for the time-dependent forced and damped harmonic oscillator (Cheng[4], 1984).

In this paper we consider the action of the form

$$S[x(t)] = \int_0^T [\frac{m}{2} (\dot{x}^2 - \omega_0^2 x^2) + fx]dt - \int_0^T dt \int_0^T G(t,s) [x(t) - x(s)]^2 ds \qquad (5)$$

with the symmetric kernel $G(t,s)$, the angular frequency ω_0 and the constant force f. The two-time quadratic term in eq. (5) is ised in the polaron problem (Feynman[5], 1955, Hellwarth and Platzman[6], 1962). It has also been considered for treating an electron gas in a random potential (Bezák[7], 1970) and in calculating the density of electronic states in disordered systems (Edwards and Guylayev[8], 1964, Sa Yakanit[9], 1979). In a recent paper (Khandekar et al[10], 1983) the authors are able to express the path integration of the action (5) with $\omega_0 = f = 0$ in terms of the solutions of certain integrodifferential equations. They then obtainthe propagator in closed form for the specific kernel Introduced by Feynman in the polaron problem. Now we are able to extend their results for ω_0 and f beIng different from zero. For convenience we will omit such derivations whlch the reader may find in the interesting papers (Khandekar et al[10], 1983; Cheng[11], 1984).

2. FORMULATION

For the action (5) we have

$$S_N = \frac{m}{2\varepsilon} \sum_{j=1}^{N} [(x_j - x_{j-1})^2 - \varepsilon^2 \omega_0^2 x_j^2] + \varepsilon f \sum_{j=1}^{N} x_j - \varepsilon^2 \sum_{j=1}^{N} \sum_{K=1}^{N} G(t_j, s_k)(x_j - x_k)^2 \qquad (6)$$

Therefore, the matrix P is of the form

$$P = L(I + L^{-1}V) \tag{7}$$

where the matrices L and V have the following eiements

$$L_{jj} = 2\left(1 + g_{jj} - \frac{\omega_0^2\varepsilon^2}{2} - \sum_{i=1}^{N} g_{ij}\right), \quad L_{j,j\pm1} = -1, \quad L \quad 0 \;\; (i\neq j, j\pm1) \tag{8}$$

and

$$V_{ij} = 2g_{ij} = V_{ji} \tag{9}$$

wlth $g_{ij} = (2\varepsilon^3/m)G(t_i, s_j)$. With the known results (Khandekar *et al*[10], 1983), we can easily show that

$$\lim_{\varepsilon\to 0}\left[\varepsilon \det(P)\right] = \lim_{\varepsilon\to 0}\left[(\varepsilon \det(L))(\det(I + L^{-1}V))\right]$$
$$= \psi(T)\exp\left\{\int^{T} dt \int \tilde{R}(t,t;\mu)\,d\mu\right\} \tag{10}$$

with the matrix $\tilde{R}(t,s;\mu) = \varepsilon^{-1}L^{-1}V + \mu L^{-1}V\tilde{R}(t,s;\mu)$ and μ a parameter. Here $\psi(T)$ and $\tilde{R}(t,s;\mu)$ are, respectively, the solutions of the following integrodifferential equations

$$\frac{1}{4}\, m\,(\ddot{\psi} + \omega_0^2\psi) + \left[\int_0^T G(t,s)\,ds\right]\psi = 0, \quad \psi(0) = 0, \quad \dot{\psi}(0) = 1 \tag{11}$$

and

$$\frac{m}{4}\left(\frac{\partial^2}{\partial t} + \omega_0^2\right)\tilde{R}(t,s) + \left[\int_0^T G(t,s)\,ds\right]\tilde{R}(t,s) = G(t,s) + \mu\int_0^T G(t,t')\,\tilde{R}(t',s)\,dt' \tag{12}$$

with

$$\tilde{R}(0,s) = \tilde{R}(T,s) \quad \text{for } 0 \leq s \leq T$$

Furthermore, we can show in the limit as $\varepsilon\to 0$, the elernents ofthe matrix U satisfy the integrodifferential equation

$$m(\ddot{u} + \omega_0^2 u) + 4\int_0^T G(t,s)\left[u(t) - u(s)\right]ds = f \tag{13}$$

wlth

$$u(0) = x_0 \text{ and } u(T) = x$$

since the matrix Y has elements $y_1 = x_0$, $y_{N-1} = x_N$ and $y_j = -\varepsilon^2 f/m$ for $2 \leq j \leq N-2$,

integrating eq. (13) over t from 0 to T, we have

$$\dot{u}(T) = \dot{u}(0) - \omega_0^2 \int_0^T u(t)\,dt + \frac{fT}{m} \tag{14}$$

since $G(t,s)$ assumed to be symmetric in t and s. We then find

$$\lim_{\varepsilon\to 0} \{\varepsilon^{-1}[(Y,U) - x_0^2 - x_N^2]\} = \lim_{\varepsilon\to 0} \{\varepsilon^{-1}(x_0 u_1 - \frac{\varepsilon^2 f}{m} \sum_{j=2}^{N-2} u_j + x_N u_N - x_0^2 - x_N^2)$$

$$= \lim_{\varepsilon\to 0} \{x_0 \dot{u}(0) - x_N \dot{u}(T) - \frac{f}{m} \int_0^T u(t)\,dt + 0(\varepsilon)\}$$

$$= (x_0 - x)\dot{u}(0) + (x\omega_0^2 - \frac{f}{m}) \int_0^T u(t)\,dt - \frac{fTx}{m} \tag{15}$$

with the help of eq. (14). Substituting (10) and (15) into (3) we obtain

$$K(x,T;x_0,0) = [m/2\pi i\hbar\psi(T)]^{1/2} \quad (\exp\{\int_0^T dt \int_{-1}^0 \tilde{R}(t,t;\mu)\,d\mu\})^{-1/2}$$

$$\times \exp\{(im/2\hbar)[(x - x_0)\dot{u}(0) - (x\omega_0^2 - \frac{f}{m}) \int_0^T u(t)\,dt + \frac{xfT}{m}]\} \tag{16}$$

The propagator is now given in terms of the solutions of the integro-differential eqs. (11)-(13). It is clear that the explicit form of the propagator can be obtained if we are able to solve them exactly.

3. EVALUATION

Considering the kernel $G(t,s)$ of the form

$$G(t,s) = \frac{m}{4} \Omega^2 \omega^2 \phi(t,s)$$

and (17)

$$\phi(t,s) = \cos[\omega(\frac{T}{2} - |t-s|)]/\,2\omega \sin(\omega T/2)$$

introduced by Feynman[5] (1955) in the polaron problem, eqs. (11) through (13) can be transformed into

$$\ddot{\psi} + \Omega_0^2 \psi = 0 \ , \ \psi(0) = 0 \ , \ \dot{\psi}(0) = 1 \tag{18}$$

$$(D^2+\Omega_0^2)\tilde{R}(t,s) - \mu\Omega^2\omega^2 \int_0^T \phi(t,t')\tilde{R}(t',s)dt' = \Omega^2\omega^2\phi(t,s), \tilde{R}(0,s) = \tilde{R}(T,s) \tag{19}$$

and

$$(D^2 + \Omega_0^2)u = \Omega^2\omega^2 \int_0^T \phi(t,s)u(s)ds + \frac{f}{m}, \; u(0) = x, \; u(T) = x \tag{20}$$

since

$$\int_0^T \phi(t,s)ds = 1/\omega^2 \tag{21}$$

Here we have set $\Omega_0^2 = \Omega^2+\omega^2$. Eq. (18) can now be easily solved and we get

$$\psi(T) = \sin(\Omega_0 T)/\Omega_0 \tag{22}$$

Applying theoperator $(D^2+\omega^2)$ on both sides of eqs.(19) and (20), we find

$$(D^2 + \Omega_1^2)(D^2 + \Omega_2^2)\tilde{R}(t,s) = \Omega^2\omega^2\delta(t-s)$$

with (23)

$$\Omega_{1,2}^2 = \dot{\lambda}_1 \mp \lambda(\mu)$$

and

$$(D^2 + \omega_+^2)(D^2 + \omega_-^2)u = \frac{\omega^2 f}{m}$$

with (24)

$$\omega_\pm^2 = \lambda_1 \pm \lambda(1)$$

respectively, where

$$\lambda_1 = (\Omega_0^2 + \omega^2)/2, \; \lambda(\mu) = \lambda = (\lambda_0^2 + \mu\Omega^2\omega^2)^{1/2}$$

and (25)

$$\lambda_0 = (\Omega_0^2 - \omega^2)/2$$

since

$$(D^2 + \omega^2)\phi(t,s) = \delta(t-s)$$

Using the known results (Khandekar et *al*[10], 1983), we show that

$$\int_{-1}^{0} d\mu \int_0^T \tilde{R}(t,t;\mu)dt = \frac{1}{2}\int_{\lambda_1}^{\lambda_0} d\lambda \quad \{\frac{T\cot[T(\lambda_1-\lambda)^{1/2}]}{(\lambda_1-\lambda)^{1/2}} - \frac{T\cot[T(\lambda_1+\lambda)^{1/2}]}{(\lambda_1+\lambda)^{1/2}}$$

$$+ \frac{1}{\lambda_1+\lambda} - \frac{1}{\lambda_1-\lambda} + \frac{(\lambda_0+\lambda)df_1/d\lambda - (\lambda_0-\lambda)df_2/d\lambda - (\lambda_0/\lambda)(f_1-f_2)}{(\lambda_0-\lambda)f_2 - (\lambda_0+\lambda)f_1} \tag{26}$$

where

$$f_j = \Omega_j[\cos(\Omega_j T) - 1]/\sin(\Omega_3 T) \tag{27}$$

Integrating eq. *(26)* and using eqs. *(25)* and (27), we obtain

$$\exp\{\int_1^0 d\mu \int^T \tilde{R}(t,t;\mu)dt\} = \frac{2\Omega_0 \; \sin(\omega_+T/2)\sin(\omega_-T/2)D}{(\omega_+^2-\omega_-^2)\sin(\Omega_0 T)\sin^2(\omega T/2)} \tag{28}$$

with

$$D = \frac{\omega_+^2 - \omega^2}{\omega_+} \sin(\omega_-T/2)\cos(\omega_+T/2) - \frac{\omega_-^2 - \omega^2}{\omega_-} \sin(\omega_+T/2)\cos(\omega_-T/2) \tag{29}$$

after lengthy but straightforward calculations.

The general solutions of eq. (24) can be expressed as

$$u(t) = a_+\sin(\omega_+t) + b_+\cos(\omega_+t) + a_-\sin(\omega_-t) + b_-\cos(\omega_-t) + \omega^2 f/m\omega_+^2\omega_-^2 \tag{30}$$

with the end-point conditions

$$u(0) = x_0 = b_+ + b_- + f/m\omega_0^2 \tag{31}$$

and

$$u(T) = x = a_+\sin(\omega_+T) + b_+\cos(\omega_+T) + a_-\sin(\omega_-T) + a_-\cos(\omega_-T) + f/m\omega_0^2 \tag{32}$$

since $\omega_+^2\,\omega_-^2 = \omega^2\omega_0^2$. Substituting eq. *(30)* into eq. (20) we find the additional conditions

$$\frac{a_+\sin(\omega_+T) - b_+[1 - \cos(\omega_+T)]}{\omega_+^2 - \omega^2} + \frac{a_-\sin(\omega_-T) - b_-[1 - \cos(\omega_-T)]}{\omega_-^2 - \omega^2} = 0 \tag{33}$$

and

$$\frac{\omega_+\{a_+[1-\cos(\omega_+T)] + b_+\sin(\omega_+T)\}}{\omega_+^2 - \omega^2} + \frac{\omega_-\{a_-[1-\cos(\omega_-T)] + b_-\sin(\omega_-T)\}}{\omega_-^2 - \omega^2} = 0 \tag{34}$$

with the help of eqs.(3.5)-(3.7) of Ref.10. Solving eqs.(31) - (34) we find

$$a_\pm = \{[\omega_\mp \sin(\omega_+T/2)\sin(\omega_-T/2) + \omega_\pm\cos(\omega_+T/2)\cos(\omega_-T/2)$$

$$\mp \frac{\omega_\pm(\omega_\mp^2-\omega^2)\sin(\omega_\mp T/2)}{(\omega_\mp^2-\omega^2)\sin(\omega_\pm T/2)}](x-x_0) \mp \frac{2\omega_\mp(\omega_+^2-\omega_-^2)\sin(\omega_+T/2)\sin(\omega_-T/2)}{\omega_\mp^2 - \omega^2}(x_0-\frac{f}{m\omega_0^2})\}/d \tag{35}$$

and

$$b_\pm = \{[\omega_\mp \sin(\omega_\mp T/2)\cos(\omega_\pm T/2) - \omega_\pm\sin(\omega_\pm T/2)\cos(\omega_\mp T/2)](x-x_0)$$

$$\mp \frac{2\omega_\mp(\omega_+^2 - \omega_-^2)\sin(\omega_\mp T/2)\cos(\omega_\pm T/2)}{\omega_\mp^2 - \omega^2}(x_0 - \frac{f}{m\omega_0^2})\}/d \tag{36}$$

where

$$d = 2(\omega_+^2-\omega_-^2)\left[\frac{\omega_+\sin(\omega_+T/2)\cos(\omega_-T/2)}{\omega_+^2 - \omega^2} \quad \frac{\omega_-\sin(\omega_-T/2)\cos(\omega_+T/2)}{\omega_-^2 - \omega^2}\right] \tag{37}$$

after straightforward calculations and simplifications. In the derivation of eqs. (35) and (36) we have assumed that $\sin(\omega_+T/2)\sin(\omega_-T/2)\neq 0$. Combining eqs. (16), (22) and (28), we finally arrive to our principal result

$$K(x,T;x_0,0) = \left[\frac{m(\omega_+^2 - \omega_-^2)\sin^2(\omega T/2)}{4\pi i\hbar D\sin(\omega_+T/2)\sin(\omega_-T/2)}\right]^{1/2} \exp\{(\frac{im}{2\hbar})\left[(x-x_0)(\omega_+a_+-\omega_-a_-)\right.$$

$$\left.-(x\omega_0^2-\frac{f}{m})\left(\frac{a_+[1-\cos(\omega_+T)]+b_+\sin(\omega_+T)}{\omega_+} + \frac{a_-[1-\cos(\omega_-T)]+b_-\sin(\omega_-T)}{\omega_-}\right)+\frac{f^2}{m^2\omega_0^2}\right]\} \tag{38}$$

with eqs. (35) and (36). For latter convenience we express the exponent in eq.(38) in terms of the coefficients of the solution (30). The explicit form of the propagator will be given in the Appendix.

4. CONCLUSION

We see that eq. (38) is invalid for $\sin(\omega_+ T/2)\sin(\omega_- T/2) = 0$. However, for the special case of $f = \omega_0 = 0$ implying $\omega_- = 0$, the integrodifferential equation (20) reads

$$(D^2 + \Omega^2)u = \Omega^2\omega^2 \int_0^T \phi(t,s)u(s)ds, \quad u(0) = x_0 , \quad u(T) = x \qquad (39)$$

and has solution

$$u(t) = \frac{(x-x_0)\Omega^2}{2\omega_+^2}\left[\cot(\omega_+ T/2)\sin(\omega_+ t) - \cos(\omega_+ t) + \frac{2\omega^2 t}{\Omega^2 T}\right] + \frac{\Omega^2(x+x_0)+2\omega^2 x_0}{2\omega_+^2} \qquad (40)$$

Taking the limit value of the normalisation factor in (38) as $\omega_- \to 0$ and using (40), we can easily show that

$$K(x,T;x_0,0) = (m/2\pi i\hbar T)^{1/2} \left[\sin(\omega T/2)/\omega\right]\left[\omega_+/\sin(\omega_+ T/2)\right]$$

$$\times \exp\left\{\frac{im\Omega^2}{2\hbar\omega_+^2}\left[\frac{\omega^2}{\Omega^2 T} + \frac{\omega_+}{2}\cot(\omega_+ T/2)\right](x-x_0)^2\right\} \qquad (41)$$

which is equivalent to (3.45) of Khandekar *et al*[10] (1983) since here $\omega_+^2 = \Omega^2 + \omega^2$. Considering now the case of $\Omega=0$ (forced harmonic oscillator), the integrodifferential equation (20) becomes

$$m(\ddot{u} + \omega_0^2 u) = f, \quad u(0) = x_0 , \quad u(T) = x \qquad (42)$$

and has the solution

$$u(t) = \left[\frac{\left[x-x_0\cos(\omega_0 T)\right] - (f/m\omega_0^2)\left[1-\cos(\omega_0 T)\right]}{\sin(\omega_0 T)}\right]\sin(\omega_0 t) + \left(x - \frac{f}{m\omega_0^2}\right)\cos(\omega_0 t) + \frac{f}{m\omega_0^2} \qquad (43)$$

After lengthy but straightforward calculations, we then obtain by our method

$$K(x,T;x_0,0) =$$

$$= \left[m\omega_0/2\pi i\hbar \sin(\omega_0 T)\right]^{1/2} \exp\{(im\omega_0/2\hbar \sin(\omega_0 T) \left[(x^2+x_0^2)\cos(\omega_0 T) - 2xx_0\right.$$

$$\left. + \frac{2(x+x_0)\left[1-\cos(\omega_0 T)\right]f}{m\omega_0^2} - \frac{2f^2}{m^2\omega_0^4}\left(\left[1 - \cos(\omega_0 T)\right] - \frac{\omega_0 T \sin(\omega_0 T)}{2}\right)\right]\} \quad (44)$$

which can easily be shown to be equivalent to eq. (3.66) for the constant force case of Ref.1, as we expect. It seems of interest to note that the exponent can be obtained much more easily from eq.(15) than to evaluate it from the Lagrangian of the system as is usually done. We agree with the authors (Khandekar *et al*[10], 1983) in that for many physical applications, the time-depenbent force term should be considered. However, from aur analysls in this paper, we see no difficulties in generalising eq. (15). The difficulties may only depend on if we are eble to solve the Integro-differential equation (20) with time-dependent force $f(t)$.

APPENDIX

With the help of eqs. (35) and (36), we can wrlte the propagator in the form

$$K(x,T;x_0,0) = \left[\frac{m(\omega_+^2 - \omega_-^2)\sin^2(\omega T/2)}{4\pi i\hbar D \sin(\omega_+ T/2)\sin(\omega_- T/2)}\right]^{1/2}$$

$$\times \exp\{\frac{im}{2\hbar d}\left[A(x - x_0)^2 + B(x^2 + x_0^2) + Cx_0 f + Ef^2\right]\} \quad (A1)$$

with

$$A = 2\omega_+\omega_-\sin(\omega_+ T/2)\sin(\omega_- T/2) + (\omega_+^2 + \omega_-^2)\cos(\omega_+ T/2)\cos(\omega_- T/2)$$

$$- \omega_+\omega_- \left\{\frac{(\omega_+ - \omega^2)\sin(\omega_- T/2)}{(\omega_-^2 - \omega^2)\sin(\omega_+ T/2)} + \frac{(\omega_-^2 - \omega^2)\sin(\omega_+ T/2)}{(\omega_+^2 - \omega^2)\sin(\omega_- T/2)}\right\} \quad (A2)$$

$$B = \frac{2\omega_+\omega_-(\omega_+^2 - \omega_-^2)\sin(\omega_+ T/2)\sin(\omega_- T/2)}{(\omega_+^2 - o^2)(\omega_-^2 - \omega^2)} \quad (A3)$$

$$C = -\frac{4\omega^2(\omega_+^2 - \omega_-^2)^2 \sin(\omega_+ /2)\sin(\omega_- /2)}{\omega_+\omega_-(\omega_+^2 - \omega^2)(\omega_-^2 - \omega^2)} \qquad (A4)$$

and

$$E = \frac{1}{2\omega_0^2}\left[1 + \frac{4\omega^2(\omega_+^2 - \omega_-^2)^2 \sin(\omega_+ T/2)\sin(\omega_- T/2)}{\omega_+\omega_-(\omega_+^2 - \omega^2)(\omega_-^2 - \omega^2)}\right] \qquad (A5$$

REFERENCES

1. R.P.Feynman and A.R.Hibbs, *Quantum Mechanics and Path Integrals*, (McGraw-Hill, New York, 1965).
2. E.W.Montroll, Commun. Pure Appl. Math, 5, 415 (1952).
3. B.K. Cheng, Rev. Bras. Fís. *13*, 220 (1983).
4. B.K. Cheng, J. Math. Phys. 25, 1804 (1984).
5. R.P.Feynman, Phys. Rev. 97, 660 (1955).
6. R.W.Hellwarth and P.M.Platzman, Phys. Rev.*128*, 1599 (1962).
7. V.Bezâk, Proc. R. Soc. A *315*, 339 (1970).
8. S.F.Edwards and V.B. Guylayev, Proc. Phys. Soc.*83*, 495 (1964).
9. Sa Yakanit, Phys. Rev. *19*B, 2267 (1979).
10. D.C.Khandekar, S.V.Lawande and K.V.Bhagwat, J.Phys.A*16*, 4209 (1983).
11. B.K. Cheng, Phys. Lett. *103*A, 357 (1984).

Resumo

Mostramos inicialmente que a integração de trajetória para o oscilador harmonico forçado com ação quadrática e com dois tempos, a qual caracteriza o efeito de memória, pode ser expressa em termos das soluções de algumas equações integro-diferenciais. A seguir, obtemos em forma fechada o propagador exato para o kernel específico introduzido por Feynman no problema do polaron.