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Nonrelativistic Exact U(1) S Matrix and Nonlinear Schrödinger Equation

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Abstract Using perturbation theory techniques we relate the $U(1)$ symmetric factorizable galilean invariant S matrices to the exact solution of the nonlinear Schrödinger equation and to some well known solutions of the Calogero system.

1. INTRODUCTION

Factorized exact S matrices form by now a well known chapter of two diniensional field theory[1,2]. However the nonrelativistic limit has not received enough attention, and our aim is to show that well established methods of quantum field theory apply as well in that limit. It has been shown' that the nonrelativistic factorized S matrix is obtained recalling that the rapidity (θ) is substitutedby nonrelativistic velocity, $\theta \to p/m$, and crossing is no more required. In this way we construct solutions with $U(1)$ symmetry which we identify by means of perturbation theory with the solution of the nonlinear Schrödinger model, and to some solutions of the Calogero system K. Sogo *et al*[3] have obtained this result as a special limit of Z_4 symmetry.

2. NONRELATIVISTIC FACTORIZED S MATRIX WITH THE U (1) SYMMETRY

We consider a field ϕ and its conjugate $+^*$, and a $U(1)$ symmetry

$$U(1)|\phi> = e^{i\beta}|\phi>, (0 \leq \beta < 2\pi) \tag{2.1}$$

The S matrix is defined as follows. We call

$$|1,\theta_1,\theta_2> = |\phi(\theta_1)\phi(\theta_2)> \tag{2.2a}$$

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$$|2,\theta_1,\theta_2> = |\phi^*(\theta_1)\phi^*(\theta_2)> \quad (2.2b)$$

$$|3,\theta_1,\theta_2> = |\phi(\theta_1)\phi^*(\theta_2)> \quad (2.2c)$$

$$|4,\theta_1,\theta_2> = |\phi^*(\theta_1)\phi(\theta_2)> \quad (2.2d)$$

so that

$$|i,\theta_1,\theta_2>_{in} = S_{ij}(\theta_1-\theta_2)|i,\theta_1,\theta_2> \quad (2.3)$$

Using charge conjugation and $U(1)$ symmetry, we see that $\hat{S}$ has the form

$$\hat{S}(\theta) = \begin{pmatrix} S_{11}(\theta) & S_{12}(\theta) & S_{13}(\theta) & S_{14}(\theta) \\ S_{21}(\theta) & S_{22}(\theta) & S_{23}(\theta) & S_{24}(\theta) \\ S_{31}(\theta) & S_{32}(\theta) & S_{33}(\theta) & S_{34}(\theta) \\ S_{41}(\theta) & S_{42}(\theta) & S_{43}(\theta) & S_{44}(\theta) \end{pmatrix} = \begin{pmatrix} S(\theta) & 0 & 0 & 0 \\ 0 & S(\theta) & 0 & 0 \\ 0 & 0 & S_r(\theta) & S_t(\theta) \\ 0 & 0 & S_t(\theta) & S_r'(\theta) \end{pmatrix} \quad (2.4)$$

We now take the factorization equations[1]

$$S(\theta)S_t(\theta+\theta')S_r(\theta') = S_t(\theta)S(\theta+\theta')S_r(\theta') + S_r'(\theta)S_r(\theta+\theta')S_t(\theta') \quad (2.5a)$$

$$S_r'(\theta)S_r(\theta+\theta')S_r'(\theta') = S_r(\theta)S_r'(\theta+\theta')S_r(\theta') \quad (2.5b)$$

$$S_r'(\theta)S_t(\theta+\theta')S(\theta') = S_t(\theta)S_r'(\theta+\theta')S_r(\theta') + S_r'(\theta)S(\theta+\theta')S_t(\theta') \quad (2.5c)$$

The above system can be easily dealt with by the usual procedure, the result for the S matrix being

$$S_r(\theta) = S_r'(\theta) \quad (2.6)$$

$$S(\theta) = \frac{\sin(\nu+b\theta)}{\sin\nu} S_r(\theta) \quad (2.7a)$$

$$S_t(\theta) = \frac{\sin b\theta}{\sin \nu} S_r(\theta) \quad (2.7b)$$

Unitarity gives us

$$S_r(\theta)S_r(-\theta) = \frac{\sin^2\nu}{\sin^2\nu - \sin^2 b} \quad (2.7c)$$

If we make $b \to 0$ and $\nu \to 0$, with A = $-b/\nu$ fixed, we get

$$S(\theta) = (1 + \lambda\theta) S_r(\theta) \tag{2.8a}$$

$$S_t(\theta) = \lambda\theta \ \ S_r(\theta) \tag{2.8b}$$

and

$$S_r(\theta) S_r(-\theta) = \frac{1}{1 - \lambda^2\theta^2} \tag{2.8c}$$

The minimal solution for $S_r(\theta)$ can be written as

$$S_r^{\min}(\theta) = - \frac{\sin \nu}{\sin b\theta} \ \frac{\Gamma(-\frac{\nu}{\pi} - \frac{b\theta}{\pi}) \ (1 + \frac{\nu}{\pi} - \frac{b\theta}{\pi})}{\Gamma(-\frac{b\theta}{\pi}) \ \Gamma(1 - \frac{b\theta}{\pi})} \tag{2.9}$$

For $b/\nu = -\lambda$ fixed while $b \to 0$ and $\nu \to 0$ we obtain

$$S_r^{\min}(\theta) = - \frac{1}{1 - \lambda\theta} \tag{2.10}$$

3. NONLINEAR SCHRÖDINGER MODEL

The nonlinear Schrödinger equation (with mass m = 1/21 can be obtained from the Lagrangian density

$$L = \frac{i}{2} \phi^* \overleftrightarrow{\partial}_0 \phi - |\partial_1 \phi|^2 - c(\phi^*\phi)^2 \tag{3.1}$$

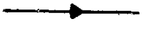
or equivalently with help of an auxiliary field a,

$$L = \frac{i}{2} \phi^* \overleftrightarrow{\partial}_0 \phi - |\partial_1 \phi|^2 + \sigma\sigma^* - \sqrt{c} \ (\phi^{*2}\sigma + \sigma^*\phi^2) \tag{3.2}$$

The Euler equation glves $\sigma(x) = \sqrt{c} \ \phi(x)^2$.

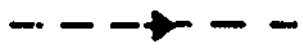
Using a perturbation theory in the parameter c we can verify that the model has no partlcle prcducêion and Its factorized S matrix is give by eqs. (2.8) and (2.10). (The calculations are analogous to those made for the massive Thirring model and sine-Gordon equation[4,5]).

The ϕ propagator Is given by



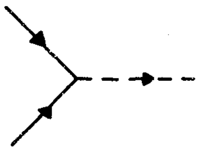
$$D^{\phi}(k) = \langle \phi(k) \phi^*(-k) \rangle = \frac{-1}{k^0 - (k^1)^2 + i\varepsilon} \tag{3.3}$$

whlle for the σ propagator we have



$$D^{\sigma}(k) = \frac{1}{1 - \dfrac{ic}{2k^0 - (k^1)^2}} \qquad (3.4)$$

and *the* vertex $\sigma^*\phi^2$ is



$$= (4ic)^{1/2} \qquad (3.5)$$

where $k = (k^0, k^1)$ is the two-dimensional momentum.

There are 3 contributions for particle pair production, which cãn be seen to cancel, namely those in fig. 1, plus permutations.

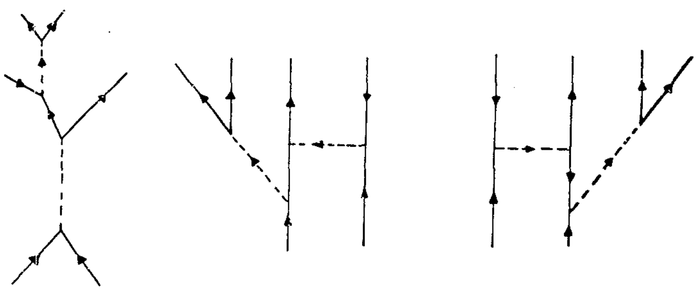
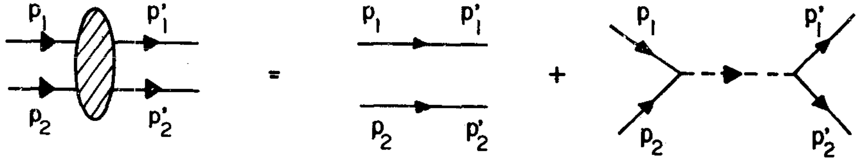


Fig.1 - The contributions for particle pair production.

The explicit S matrix is shown in fig. 2 and is written

$$\langle p_1' p_2' | S | p_1 p_2 \rangle = \langle p_1' p_2' | p_1 p_2 \rangle + (2\pi)^2 \; \delta(p_1' + p_2' - p_1 - p_2) 4ic \; D^{\sigma}(p_1 + p_2) \qquad (3.6)$$



Fiy. 2 - Graphs of the ϕ-particle S matrix.

Substituting the expression for the propagator, taken from eq. (3.4), eq. (3.6) can be written in a simple form

$$\langle p_1'p_2'|S|p_1p_2\rangle = \frac{1 + ic/(p_1^1-p_2^1)}{1 - ic/(p_1^1-p_2^1)} \langle p_1'p_2'|p_1p_2\rangle \tag{3.7}$$

If we recall that

$$p_1^1 - p_2^1 = m(\theta_1-\theta_2) = \theta/2 \tag{3.8}$$

we obtain the same form as the exact solution (2.8) by choosing

$$\lambda = -im/c = -i/2c \tag{3.9}$$

As already mentioned by Zamolodchikov and Zamolodchikov[1], the S matrix (2.9) can be identified with the S matrix for that of Calogero system[6]. In that case the potentials are given by Weierstrass's -functions[7]. Solution (2.9) describes the scattering by a Pöschl-Teller potential whose complete solution can be found in ref. 8.

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Resumo

Usando técnicas da teoria de perturbação, relacionamos as matrizes S com simetria $U(1)$ com a matriz S exata do modelo de Schrödinger não linear e com as matrizes S de algumas soluções do sistema de Calogero.