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Wilson Loop Interaction at Finite Temperature

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Abstract In this note we examine some features of long-range interactions between electrically neutral systems represented by rectanyular Wilson loops in the presence of a heat reservolr. The temperature independente of the interaction is obtained.

1. INTRODUCTION

The analysls of the interaction of neutral colour states in non-abelian quantum gauge theories at zero temperature has revealed the existence of long-range forces, like van der Waals forces in atomic and molecular physics[1,2]. On the other hand, it is well-known that the introduction of a heat reservoir can modify the zero-temperature physical phenomena[3].

In this note we analyse these long-range Interactions in the simple case of a quantized electromagnetic field in contact wlth a heat reservolr by coniputing the interaction of electrlcally neutral systems represented by rectangular Wilson loops.

Our concluslon concerns the temperature Independence of these long-range forces.

This note is planned as follows: In Section 2 we compute the dipole-dipole Wilson loop Interaction and the static potentlal at zero temperature. In Section 3 we analyse the effects of a heat reservoir on the interactions.

2. WILSON LOOP EVALUATION AT ZERO-TEMPERATURE

We consider a neutral system simulated as an external current circulating around a rectangle $C_{(R,T)}$ as shown in Fig. 1.

The interactlon energy between two such neutral sources separated by a space-like distance h is computed by evaluating the vaccum

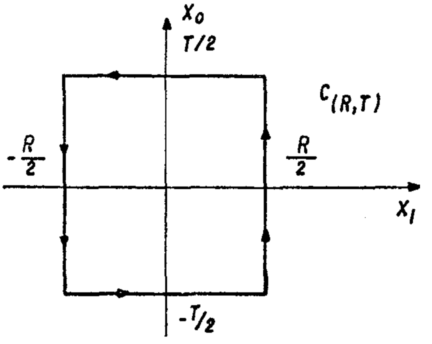

Fig.1 - The rectangle $C_{(R,T)}$ used to define the Wilson Loop.

energy of the quantized electromagnetic field in the presence of these sources and then subtracting off their self-energies

$$E(h) = \lim_{T\to\infty} - \frac{1}{2T} \log \left[\frac{<\exp(ie \oint_{C^{(1)}_{(R,T)}} A_\mu dx_\mu)\exp(ie\oint_{C^{(2)}_{(R,T)}} A_\mu dx_\mu)>}{<\exp(ie \oint_{C^{(1)}_{(R,T)}} A_\mu dx_\mu)><(\exp(ie \oint_{C^{(2)}_{(R,T)}} A_\mu dx_\mu)>} \right] \tag{1}$$

where the rectangle $C^{(2)}_{(R,T)}$ is translated through the distance h from the rectangle $C_{(R.T)}$ along its spatial direction. The factor 2 in eq. (1) prevents the double counting of the interaction energy.

The quantum average < > in eq. (1) is defined by the Euclidean generating functional of the quantized electromagnetic field

$$<O(A_\mu)> = \int \mathcal{D}[A_\mu(x)]\exp(- \frac{1}{4} \int d^D x F^2_{\mu\nu}) O(A_\mu) \tag{2}$$

where $O(A_\mu)$ denotes an observable, and $\mathcal{D}[A_\mu(x)]$ the appropriately normalized functional measure ($<1> = 1$) including gauge fixing terms. We call attention to the usefulness of the representation of neutral objects by Wilson loops, since eq.(1) manifestly exhibits the gauge-invariance of the calculation.

In order to evaluate eq.(1) it is convenient to express the Wilson Loops by means of external currents $J_\mu(x;\ C^{(i)}_{(R,T)})$ circulating around the contours $C^{(i)}_{(R,T)}$ parametrized by $x^{(i)}_\mu = x^{(i)}_\mu(s)$ with $i=1,2$[6,7]

$$J_\mu(x;\ C^{(i)}_{(R,T)}) = ie \oint_{C^{(i)}_{(R,T)}} \delta^{(D)}(x_\mu - x^{(i)}_\mu(s))dx^{(i)}_\mu(s) \quad (\mu = 0,1;\ldots;D-1) \tag{3}$$

The Interaction energy in eq. (1) can be exactly evaluated, as the euclidesn functional integrais involved are of the Gaussian type, thus giving the following result

$$E(h) = \lim_{T\to\infty} -\frac{1}{T}\log\left[\exp\{\frac{1}{2}\int d^Dx d^Dy J_\mu(x;C^{(1)}_{(R,T)})\Delta^{(E)}_{\mu\nu}(x-y)J_\nu(y;C^{(2)}_{(R,T)})\}\right] \tag{4}$$

where we have chosen the Feynman gauge and

$$\Delta^{(E)}_{\mu\nu}(x-y) = \delta_{\mu\nu}\int \frac{d^Dk}{(2\pi)^D}\; e^{-ik.(x-y)}\cdot\frac{1}{k^2}$$

means the associated (Eucl idean) Feynman propagator.

The evaluation of eq. (4) can be accompl ished by writing it in momentum spece

$$E(h) = \lim_{T\to\infty} -\frac{1}{2T}\left[\int \frac{d^Dk}{(2\pi)^D} f_\mu(k;\ C^{(1)}_{(R,T)})\ \frac{\delta_{\mu\nu}}{k^2}\ f_\nu(-k;\ C^{(2)}_{(R,T)})\right] \tag{5}$$

with

$$f_\mu(k;\ C^{(i)}_{(R,T)}) = ie\oint_{C^{(i)}_{(R,T)}} e^{-ik_\alpha . x_\alpha(s)}\ dx_\mu(s)\ ,\ (\alpha,\mu = 0,1,\ldots D-1) \tag{6}$$

As the rectangles $C^{(i)}_{(R,T)}$ are contained in a two-diinensional sub -space of ttie space-time R^D, we can decompose the vector $\vec{k}$ as $\vec{k} = k_0\vec{e}_0 + k_1\vec{e}_1 + \hat{k}$, where $\hat{k}$ is the projection of $\hat{k}$ over the sub-space perpendicular to the sub-space $\{\vec{e}_0;\vec{e}_1\}$ containing $C^{(i)}_{(R,T)}$. In addition, the space coordinate system is chosen so that the x -axls direction coincides with the one defined by the spatial sides of . the rectangles $C^{(i)}_{(R,T)}$. This coordinate choice implies the validity of the following relatlons between the contour-functionals in eq. (6)

$$f_0(k;\ C^{(2)}_{(R,T)}) = e^{-ik_1.h} f_0(k;C^{(1)}_{(R,T)})$$

and

$$f_1(k;\ C^{(2)}_{(R,T)}) = e^{-ik_1.h} f_1(k;C^{(1)}_{(R,T)}) \tag{7}$$

A simple evaluation of eq.' (6) provides the solutions

$$f_0(k;C^{(1)}_{(R,T)}) = -\frac{4e}{k_0}\sin(\frac{k_0T}{2})\sin(\frac{k_1R}{2})$$

and

$$f_1(k;C^{(1)}_{(R,T)}) = \frac{4e}{k_1}\sin(\frac{k_0 T}{2})\sin(\frac{k_1 R}{2}) \tag{8}$$

Inserting eqs. (7) and (8) into eq. (5), we obtain

$$E(h) = \lim_{T\to\infty} \frac{8e^2}{T}\left\{\int_{-\infty}^{+\infty}\frac{dk_1}{(2\pi)}e^{-ik_1.h}\frac{\sin^2(\frac{k_1 R}{2})}{k_1}\right. \tag{9}$$

$$\left.\left[\int\frac{d^{D-2}\hat{k}}{(2\pi)^{D-2}}\left(\int_{-\infty}^{+\infty}\frac{dk_0}{(2\pi)}\frac{(k_0^2+k_1^2)}{k_0^2}\frac{1}{k_0^2\,k_1^2\,\hat{k}^2}\sin^2(\frac{k_0 T}{2})\right)\right]\right\}$$

The integration in k_0-variable is easily performed by using the formulas 3.824-1 and 3.826-1 from Ref. 8. After taking the limit $T\to\infty$, we get

$$E(h) = 2e^2\left[\int_{-\infty}^{+\infty}\frac{dk_1}{(2\pi)}e^{-k_1.h}\sin^2(\frac{k_1.R}{2})\left(\int\frac{d^{D-2}\hat{k}}{(2\pi)^{D-2}}\frac{1}{(k_1^2+\hat{k}^2)}\right)\right] \tag{10}$$

In order to calculate eq.(10) we use the dimensional regularization scheme[9]. By making use of the relation (3.8) from Ref. 9 (analytically continued to Euclidean spãce-time) we can perform the integration in k-variable

$$E(h) = -\frac{e^2\Gamma(2-\frac{D}{2})}{2^{D-1}\pi^{(D-2)/2}}\left[\int_{-\infty}^{+\infty}\frac{dk_1}{(2\pi)}|k_1|^{D-4}\left[e^{-ik_1(R+h)}+e^{ik_1.(R-h)}-e^{-ik_1 R}\right]\right]$$

(11)

The Fourier transforms in eq. (11) are tabulated[10] in the form

$$\int_{-\infty}^{+\infty} e^{-i\alpha.x}|x|^{\beta}dx = -2\sin\frac{\beta\pi}{2}\cdot\Gamma(\beta+1)\;|\alpha|^{-\beta-1} \tag{12}$$

Finally, we obtaln the expression for the interaction energy between Wilson loops at zero temperature

$$E(h) = \frac{e^2\Gamma(2-\frac{D}{2})\Gamma(D-3)}{2^{D-1}\cdot\pi^{D/2}}\sin((D-4)\frac{\pi}{2})\{(h+R)^{-D+3}+(h-R)^{D+3}-2h^{-D+3}\}$$

(13)

In order to study eq.(13) for the physical limit $D=4$ we note that the pole of the gama-function $\Gamma(2-\frac{D}{2})$ cancels the zero of the sine function $\sin(D-4)\frac{\pi}{2}$, namely

$$\lim_{D\to 4} \Gamma(2-\frac{D}{2})\sin(D-4)\frac{\pi}{2} = -\pi \qquad (14)$$

which provides the four-dimensional interaction energy as a multipole expansion

$$E(h) = -\frac{e^2}{8\pi}\{(h+R)^{-1}+(h-R)^{-1}-2.h^{-1}\} = -\frac{e^2}{4\pi}\left(\sum_{k=1}^{\infty}\frac{R^{2h}}{h^{2k+1}}\right) \qquad (15)$$

From eq.(15) we readily observe that the dominant term in the asymptotic limit $h\to\infty$ comes from the classical Jipole-dipole interaction. Furthermore, the interaction is attractive since the dipolar rnoments of the neutral systems analysed are parallel.

For completeness we have evaluated the static potential of two sources (see sect. VI Ref. (4)) using Wilson loops and the dimensional regularization scheme, which is

$$V(R) = \lim_{T\to\infty} -\frac{1}{T}\log\langle\exp(ie\oint_{C_{(R,T)}} A_\mu dx_\mu)\rangle \qquad (16)$$

The result yields the $(D-1)$ *dimensional Coulomb Zaw*

$$V(R) = \frac{e^2\Gamma(2-\frac{D}{2})}{\pi^{D/2}\,2^{D-2}}\sin\left[\frac{(D-4)\pi}{2}\right]\Gamma(D-3)\;|R|^{-D+3} \qquad (17)$$

where we have used the dimensional regularization rule which assigns the value zero to the *tad pole* R-independent integral

$$\frac{e^2\Gamma(2-\frac{D}{2})}{2^{D-2}\pi^{(D-2)/2}}\int_{-\infty}^{+\infty}\frac{dk_1}{(2\pi)}\,|k_1|^{D-4} = 0 \qquad (18)$$

The usual Coulornb law in three dimensions is obtained by taking the physical limit $D\to 4$ in eq. (17), resulting $V(R) = -e^2/4\pi R$.

3. WILSON LOOP EVALUATION AT NON-ZERO TEMPERATURE

We now examlne the presence of a heat reservoir at temperature $T = 1/k_B\beta$ (k_B is the Boltzmann's constant) in the quantum gauge system.

We first evaluate the free energy of two static sources[3]

$$V(R;\beta) = -\frac{1}{\beta}\log\langle \exp ie\oint_{C_{(R,\beta)}} A^{\beta}_{\mu}(x)\, dx_{\mu} \rangle \qquad (19)$$

where now the rectangle $C_{(R,\beta)}$ has its temporal sides extending from 0 to β. The quanturn average < > involved in eq. (19) is defined by the Euclidean partition functional of the quantized electromagnetic field at ternperature T [11]

$$\langle O(A^{\beta}_{\mu}(x))\rangle = \int \mathcal{D}[A^{\beta}_{\mu}(x)]\ \exp\{-\frac{1}{4}\int_0^{\beta} d\tau \int d^{D-1}\vec{x}(F_{\mu\nu})^2\}.O(A^{\beta}_{\mu}(x)) \qquad (20)$$

Here $\mathcal{D}[A^{\beta}_{\mu}(x)]$ means the normalized functional rneasure over all thermal gauge fields $A^{\beta}_{\mu}(x)$ satisfying the periodicity condition

$$A^{\beta}_{\mu}(\vec{x},0) = A^{\beta}_{\mu}(\vec{x},\beta) \qquad (21)$$

A convenient interpretation for eqs. (19) and (21) consists in considering that at finite temperature the space-time possesses the topology of a cilinder $S^1 \times R^{D-1}$ instead of the usual topology R^D.

The periodicity conditions in eq.(21) imply that the Wilson loop contour integration around $C_{(R,\beta)}$ is reduced to the contour integration along their temporal sides only, i.e.

$$\exp(ie\oint_{C_{(R,\beta)}} A^{\beta}_{\mu}(x)dx_{\mu}) = \exp(ie\int_0^{\beta} A^{\beta}_{0}(0,\tau)d\tau)\ \exp(-ie\int_0^{\beta} A^{\beta}_{0}(R,\tau)d\tau) \qquad (22)$$

In order to evaluate eq. (19) we express the *Wilson Strings* in eq. (22) by means of external localized currents

$$\tilde{J}_0(\vec{x},\tau) = ie[\delta^{(D-1)}(\vec{x}) - \delta^{(D-1)}(\vec{x}-\vec{R})] \qquad (23)$$

$$\tilde{J}_i(\vec{x},\tau) = 0 \qquad (i = 1;\ldots D-1)$$

and compute the Gaussian functional integration, yielding the result

$$V(R,\beta) = -\frac{1}{\beta}\{\frac{1}{2}\int_0^{\beta}d\tau\int_0^{\beta}d\tau'\int d^{D-1}\vec{x}\, d^{D-1}\vec{y}\ \tilde{J}_{\mu}(\vec{x},\tau)\Delta^{(E)}_{\mu\nu}(\vec{x}-\vec{y},\tau-\tau';\beta)\tilde{J}_{\nu}(\vec{y},\tau')\} \qquad (24)$$

where $\Delta^{(E)}_{\mu\nu}$ (&$,r-r' ,β) denotes the thermal Euclidean Feynman propagator in the Feynman gauge[11], namely

$$\Delta^{(E)}_{\mu\nu}(x-y,\tau-\tau',\beta) = \frac{1}{\beta}\sum_{n=-\infty}^{+\infty} \delta_{\mu\nu} \int \frac{d^{D-1}\vec{k}}{(2\pi)^{D-1}} \frac{e^{i\vec{k}.(x-y) + i\omega_n(\tau-\tau')}}{\vec{k}^2 + \omega_n^2} \tag{25}$$

$$\left(\omega_n = \frac{2\pi n}{\beta};\ n \in \mathbb{Z}\right)$$

With the orthogonality relations

$$\int_0^\beta d\tau \int_0^\beta d\tau'\ e^{i\omega_n(\tau-\tau')} = \begin{cases} 0 & n\neq 0 \\ \beta^2 & n\neq 0 \end{cases} \tag{26}$$

which suppress the modes with $\omega_n \neq 0$ in eq.(25), we simplify eq.(24) to the form

$$V(R,\beta) = e^2 \int \frac{d^{D-1}\vec{k}}{(2\pi)^{D-1}} \frac{(1 - \cos \vec{k}.\vec{R})}{\vec{k}^2} \tag{27}$$

We observe in eq. (27) the temperature independence of the free energy. Now it is convenient to chose the k_1-axis along the direction of the vector R. We thus obtain the result

$$V(R,\beta) = -e^2 \left[\int \frac{dk_1}{(2\pi)} \frac{1}{2} (e^{ik_1R} + e^{-ik_1R}) \int \frac{d^{D-2}\hat{k}}{(2\pi)^{D-2}} \frac{1}{(k_1^2+\hat{k}^2)} \right] \tag{28}$$

which is evaluated as before (see eqs. (11) and (12)), giving

$$V(R,\beta) = \frac{e^2}{2^{D-2}\pi^{D/2}} \Gamma(2 - \frac{D}{2})\Gamma(D - 3)\sin((\frac{D-4}{2})\pi) . |R|^{-D+3} \tag{29}$$

From eq. (29) we notice its coincidence with the electrostatic potential at zero temperature. (See eq. (17)).

Finally, we evaluate the free energy of the previous Wilson loops simulating neutral objects in contact with a heat reservoir at temperature T.

The evaluation of eq.(1) is now performed by means of the quantum average furnished in eq. (20) and its result in coordinate space reads

$$E(h;\beta) = -\frac{1}{\beta}\{-\frac{e^2}{2}\int_0^\beta d\tau \int_0^\beta d\tau' \int d^{D-1}\vec{x}\, d^{D-1}\vec{y}$$

$$[\delta^{(D-1)}(\vec{x}) - \delta^{(D-1)}(\vec{x}-\vec{R})]\,\Delta^{(E)}_{\mu\nu}(\vec{x}-\vec{y},\tau-\tau',\beta)\ [\delta^{(D-1)}(\vec{y}-\vec{h}) - \delta^{(D-1)}(\vec{y}-\vec{R}-\vec{h})]\} \tag{30}$$

Writing eq. (30) in momentum space we obtain the expressicn

$$E(h;\beta) =$$

$$= -\frac{1}{2}e^2\{\int_{-\infty}^{+\infty}\frac{dk_1}{(2\pi)}\int\frac{d^{D-2}\vec{k}}{(2\pi)^{D-2}}\frac{1}{(\hat{k}^2+k_1^2)}\,(e^{-k_1.(R+h)} + e^{-k_1.(R-h)} - 2e^{-ik_1.R})$$

$$= \frac{e^2\ \Gamma(2-\frac{D}{2})\ \Gamma(D-3)}{2^{D-1}\ \pi^{D/2}}\sin[(D-4)\frac{\pi}{2}]\ \{(h+R)^{-D+3} + (h-R)^{-D+3} - 2h^{-D+3}\} \tag{31}$$

The above result clearly shows that the free energy interaction of neutral systems represented by rectangular Wilson loops is temperature Independent and turns out to be of the same form as the corresponding quantity in the zero temperature regime (see eq. (13)).

We now make some concluding remark on the results in eqs.(29) and (31). We understand that these results imply that to detect the temperature effects in the interactions analysed above, one should consider the matter fields in the quantum system (quantum electrodynamics) since in this case the radiative corrections induced on the N-point photon propagator are temperature dependent, which results in a renormalized temperature dependent electronic charge $e(R;\beta)$ in the interactions in eqs. (29) and (31).

We intend in a forthcoming paper to analyse the introduction of the matter fields in the interactions analysed in this note.

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Resumo

Analisamos nesta nota a interação entre sistemas eletricamente neutros representados por Loops de Wilson retangulares na presença de um reservatório térmico.