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Instantons and Non-Abelian Magnetic Monopoles

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Abstract Certain topological relations between instantons and magnetic rnonopoles are presented. In particular,it is shown that the numbers of zero mode wave functions with definite chirality are the sarne for a given isospin representation.

1. INTRODUCT'ION

In this paper we present certain relations between the vacua sector of a non abelian gauge field theory, Yang-Mills theory, and the magnetic monopole sector of a theory described by an non abelian gauge field and a Higgs field that presents spontaneous symrnetry breaking. These relations are consequence of topological properties of the non abelian gauge field A_μ of the Yang-Mills theory defined in the R^4 euclidean space, which can be extended to the magnetic rnonopole field A_i $(i = 1,2$ and $3)$.

In section 2, we show that the gauge invariant monopole field, which is denoted by $|M,\theta\rangle$, does not present a defined chirality, and depends of the parameter θ which can vary between 0 and 2π. We also show in this section that the chirality non conservation in the $|M,\theta\rangle$ field is large, i.e. of $O(1)$.

In section 3, we show that the numbers of fermionic zero modes with definecl chirality, solutions of the Dirac in presence of the monopole field with topological charge, are the same for a given isospin representation.

2. $|M,\theta\rangle$ STATE (Chirality Breaking)

The 't Hooft-Polyakov[1] magnetic monopole solutions are obtained by considering a Lagrangean with SU(2) gauge symmetry broken, by Higgs

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mechanics, to $U(1)$, namely

$$L = L_{YM} = L_H = \frac{1}{2g^2}\,\mathrm{Tr}\,F_{\mu\nu}F_{\mu\nu} + \frac{1}{4}\,\mathrm{Tr}(D_\mu\phi)^2 + \lambda(\mathrm{Tr}\phi^2 - 2c^2) \tag{1}$$

where we have used the standard notation.

The magnetic monopole solution of eq. (1) is given by

$$\phi^{cl}(\vec{x}) = c(1-H(r))\hat{x}.\vec{\tau} \tag{2a}$$

$$A_i^{cl}(\vec{x}) = \epsilon_{aij}\,\hat{x}_j\,\frac{(1-K(r))}{r}\,\frac{\tau a}{2i} \tag{2b}$$

$$A_0^{cl}(\vec{x}) = 0 \tag{2c}$$

where $r = \sqrt{\vec{x}^2}$, and $\hat{x}_i = x_i/r$; $H(r)$ and $K(r)$ are functions that obey the following boundary conditions

$$H(0) = K(0) = 1 \quad , \quad H(\infty) = K(\infty) = 0 \tag{3}$$

$H(r)$ and $K(r)$ decreases exponentially for large r, i.e., for $r \gg c^{-1}/\lambda$ and. $r \gg c^{-1}/g$.

The topological charge q, the Chern number, of an anti-hermitian gauge field configuration defined in the R^4 euclidean space with a positive defined metric and $t = x_4$ is

$$q = -\frac{1}{16\pi^2}\int d^3r \int dt\ \mathrm{Tr}\,F_{\mu\nu}\tilde{F}_{\mu\nu} \tag{4a}$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - [A_\mu, A_\nu] \tag{4b}$$

and dual tensor

$$\tilde{F}_{\mu\nu} = 1/2\ \varepsilon_{\mu\nu\lambda\rho}F_{\lambda\rho} \tag{4c}$$

For the configuration (2a-c), we have $\mathrm{Tr}\,F_{\mu\nu}\tilde{F}_{\mu\nu} = -2E_i^a B_i^a = 0$ because $E_i^a = 0$, and its topological charge is $q = 0$. We represent this state, with a magnetic charge and zero topological charge, by $|M,0\rangle$.

We shall show that, by a convenient gauge transformation in $A_i^{cl}(x)$, which leaves the Higgs sector invariant, a monopole-like solution, with finite energy and non trivial topological charge, is obtained. The new field configuration is

$$A_i = g(x,t)A_i^{cl}g^{-1}(x,t) + g(x,t)\partial_i g^{-1}(x,t) \tag{5a}$$

$$\phi = g(x,t)\ \phi^{cl}\ g^{-1}(x,t) = \phi^{cl} \tag{5b}$$

where $g(x,t)$ is an element of the SU(2) gauge group wich leaves the Higgs field invariant.

In this new field the temporal gauge is prererved, i.e., $A_0^{cl}=0$. The above field configuratlons are equivalent to a gauge

$$A_i(x) = A_i^{cl}(x)$$

$$A_0(x,t) = g^{-1}(x,t)\partial_t\, g(x,t)$$

$$\phi(x) = \phi^{cl}(x)$$

Let us define the object

$$n[g] = \frac{-1}{24\pi^2}\int d^3x\ \varepsilon_{ijk}\ \mathrm{Tr}\left[g(x)\ \partial_i\ g^{-1}(x)g(x)\partial_j\ g^{-1}(x)g(x)\partial_k g^{-1}(x)\right] \tag{6}$$

which characterizes the "local topological charge", or "winding number", of an SU(2) gauge group element $g(\vec{x})$. It is known[2] that, if $g(x)$ is $\hat{x}$ independent when r tends to infinite, then $n[g]$ is an integer. The number $n[g]$ labels the different homotopy classes of mapping made by $g(\hat{x})$ in R^3 space, with all infinite points identified.

With eq. (6) defined, we can calculate the topological chargeof the gauge field configuration A_i, eq. (5a)

$$A_i(x,t) = g(x,t)A_i^{cl}\ g^{-1}(x,t) + g(x,t)\partial_i\ g^{-1}(x,t) \tag{7}$$

We know that

$$\mathrm{Tr}\ F_{\mu\nu}\ \tilde{F}_{\mu\nu} = \partial_\mu\ \ I_\mu \tag{8a}$$

where

$$I_\mu = \varepsilon_{\mu\nu\lambda\rho}\ \mathrm{Tr}\left[A_\nu\ F_{\lambda\rho} - (\tfrac{2}{3})\ \ A_\nu\ A_\lambda\ A_\rho\right] \tag{8b}$$

Since the gauge fields eq. (7) are regular, we can transform the volume integral into to a surface integral, via Gauss theorem. Then

$$q = -\frac{1}{16\pi^2}\int d\sigma_\mu\, I_\mu \tag{9a}$$

If we choose the surface to be an infinite hyper-cylinderasin fig.1, we have

$$q = \frac{-1}{16\pi^2}\left\{\int_I d^3x\, I_0\Big|_{t=\infty} - \int_{III} d^3x\, I_0\Big|_{t=-\infty} + \int_{\infty}^{\infty} dt \lim_{r\to\infty}\int d\delta i\, I_i\right\} \tag{9b}$$

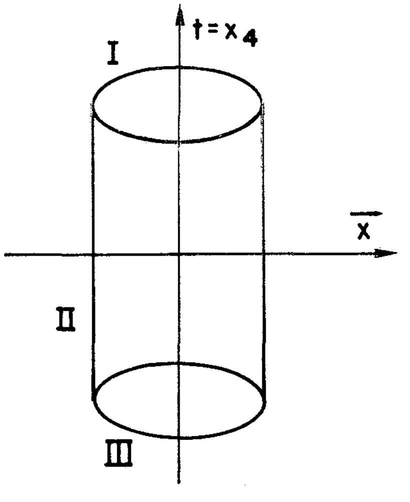

Fig.1 - Infinite hyper-cylinder used to define q in eq. (9b).

For the configuration (7a-b), we obtain

$$q = n[g(t=\infty)] - n[g(t=-\infty)] \tag{9c}$$

Let the gauge $g(\vec{x},t)$ be such

$$\lim_{t\to\infty} g(\vec{x},t) = g(x) \quad \text{and} \quad \lim_{t\to-\infty} g(\vec{x},t) = I$$

In this case we have

$$q = n[g(t=\infty)] = n[g] \tag{9d}$$

If we choose the particular gauge below

$$g(\vec{x}) = g_1(\vec{x}) = \frac{\lambda^2-r^2}{\lambda^2+r^2} + 2i\,\frac{\lambda\vec{\tau}.\vec{x}}{\lambda^2+r^2} \tag{10}$$

which is such that $g_1(r=0)=I$, and $g_1(r\to\infty)=-I$, we have for an arbitrary positive parameter λ, that

$$n[g_1] = 1 \tag{11}$$

and then $q=1$.

We can also write

$$g_1(x) = \exp[2iW(r)\hat{x}.\vec{\tau}/2] \tag{12a}$$

where

$$W(r) = \text{arc tg}\left(\frac{-2\lambda r}{r^2-\lambda^2}\right) \tag{12b}$$

The total, normalized, action of our system[3], with $E_{mono} = 0$ is

$$S_{A,\phi} = \int dt' \{\left(d^3x\ L - M_{mono}\right)\} \tag{13}$$

To evaluate eq. (13), we would use the regular representation for the magnetic monopole field, i.e. $K(r) = \theta(r_M - r)$, where r_M is the monopole radius, cf order c^{-1}. In this paper, we use the fact that the parameter c in eq. (1) is large, and then r_M is an infinetesimal quantity. For an spherically syrnmetrical gauge $g(x,t) = \exp(2iW(r,t)\vec{x}\,.\,\vec{\tau}/2)$ we have

$$S_{A,\phi} = \frac{8\pi}{g^2}\int_{\infty}^{\infty} dt \int_0^{\infty} dr(\partial_r\partial_t W(r,t))^2\ r^2 \tag{14}$$

If we choose

$$W(r,t) = W(r)f(t)$$

(notice that since $W(0,t) = 0$, then, $g(0,t) = I$), where $f(t=-\infty)=0$ and $f(t=\infty)=1$, this particular gauge field yields $q=1$, and

$$S_{A,\phi} = 8\pi/g^2 \int_{\infty}^{\infty} dt\ (\dot{f}(t)^2\ dt \int_{\infty}^{\infty} dr(W(r))^2 r^2$$

With a convenient form

$$f(t) = (1+e^{-t/\rho})^{-1}$$

where ρ is a mass^{-1} scale, then

$$S_{A,\phi} = (4\pi^2/3\,g^2)\ \frac{\lambda}{\rho}$$

Defining a dlmensionless parameter $\lambda' = \lambda/\rho$ this action is a quantity arbitrarily small for small λ' ($S = O(\lambda')$).

It is possible to construct an state $|M,1\rangle$, with magnetic charge, and topological charge equal to one, From the state $|M,0\rangle$, via a convenient gauge transformation $g(\vec{x},t)$, without a significant suppresslon factor. Let us remark that for instantons, i.e. in the vacuum sector, the transition amplitude is supressed by $e^{-8\pi^2/g^2}$ [4]. Thus we have

$$|M,1\rangle = [g]\ |M,0\rangle \tag{15a}$$

Repeated application of $g_1(x)$, i.e. applying

$$g_r(x) = [g_1(x)]^r$$

($r \in Z$), to $|M,0\rangle$ produces a state with a topological charge equal to $q=n[g_r]=r$, i.e.

$$|M,r\rangle = [g]^r|M,0\rangle \tag{15b}$$

The gauge invariant state will be obtained from a linear combination of $|M,r\rangle$ states (see ref. (5) for the instantons case), i.e.

$$|M,\theta\rangle = \sum_{r=-\infty}^{\infty} e^{-ir\theta}\ [g]^r\ |M,0\rangle \tag{16}$$

This state does not have a deflnite chirality.

The above result suggest that, in the monopole sector, the chirality is not conserved, and the amplitude For this non conservation is large, of order $O(1)$.

Note: we can construct gauge field configurations, by the same method used above, wlth arbitrary topological charge, semi-integer in particular, without an infinite action. This is because the boundary condition for $W(r,t)$ is open. For $q = 1/2$, it is only necessary that $W(r=\infty,\ t=\infty)=\pi/2$.

3. ZERO MODES

Massless ferrnions can be included in the system under study simply by adding to eq. (1) the term

$$L_F = \bar{\psi}(i\not{D})\psi \tag{17}$$

where

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \quad \text{and} \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \tag{18a,b}$$

and

$$\not{D} = \not{\partial} - i\not{A} \tag{18c}$$

The gauge field A_μ presents, in this case, a non trivial topological charge. By the Atyah-SInger index theorem, we can obtain information about the zero mode waves, and their respective chirality.

For a $|M,r\rangle$ state, the numbers of zero modes with positive chirality (n_+), and with negative chirality (n_-) satisfy the following relation

$$n_- - n_+ = \frac{1}{16\pi^2} \int d^4x \, Tr \, F_{\mu\nu} \tilde{F}_{\mu\nu} \tag{19a}$$

We are using the notation $A_\mu = A^a_\mu T^a$, where T^a is a representation of the isospin group. We have

$$\text{Tr } T_a T_b = (\tfrac{1}{3}) \, T(T+1)(2T+1)\delta ab$$

where T is the total isospin of the fermion field. Then

$$n_- - n_+ = (\tfrac{2}{3}) \, T(T+1)(2T+1)r \tag{19b}$$

For $T_a = \tau_a/2$, $T = 1/2$, and $n_- - n_+ = r$.

Jackiw and Rebbi[6] have shown for an Instanton (anti-instanton) gauge field configuration, r=1 (r=-1), that n_+=0 (n_-=0), i.e., there is nozeromodefermionwavewithpositive (negative) chirality. We now show that this result is also valid for any non-abelian monopole field with a non-trivial topological charge. (We continue to use theeuclidean metric). The Dirac equation is in this case

$$[\gamma_4(\partial_4 - A_4) + i\gamma_i(\partial_i - iA_i)]\psi = 0 \tag{20a}$$

or

$$[\tilde{\gamma}_\mu(\partial_\mu - iA_\mu)]\psi = 0 \tag{20b}$$

where

$$\tilde{\gamma}_4 = \gamma_0, \; \gamma_i = i\gamma_i, \; [\tilde{\gamma}_\mu, \tilde{\gamma}_\nu]_+ = 2\delta_{\mu\nu} \tag{20c}$$

The fermion field may be decomposed as

$$\psi_{\pm} = \begin{pmatrix} \chi_{\pm} \\ \pm \chi_{\pm} \end{pmatrix} \tag{21a}$$

where $\pm$ refer to the helicity, and χ is a 2×2 matrix, one index refering to spin, and the other to isospin. The matrix equation is

$$\begin{pmatrix} (\partial_4 - iA_4) & i\sigma_i(\partial_i - iA_i) \\ -i\sigma_i(\partial_i - iA_i) & -(\partial_4 - iA_4) \end{pmatrix} \begin{pmatrix} \chi_{\pm} \\ \pm\chi_{\pm} \end{pmatrix} = 0 \tag{21b}$$

We obtain from eq. (21b) two independent equations

$$[(\partial_4 - iA_4) + i\sigma_i(\partial_i - iA_i)]\chi_+ = 0 \tag{22a}$$

and

$$[(\partial_4 - iA_4) - i\sigma_i(\partial_i - iA_i)]\chi_- = 0 \tag{22b}$$

Defining the matrices $\bar{\alpha}_\mu$ and $\alpha_\mu = \bar{\mu}^*_\mu$, whrre $\bar{\alpha}_\mu = (i\sigma_i, I)$, we may rewrite the above equation in a compâct notation

$$\bar{\alpha}_\mu(\partial_\mu - iA_\mu)\chi_+ = 0 \tag{23a}$$

$$\alpha_\mu(\partial_\mu - iA_\mu)\chi_- = 0 \tag{23b}$$

The matrices $\bar{\alpha}_\mu$ and a_μ satisfy the following relations

$$\bar{\alpha}_\mu \alpha_\nu = \delta_{\mu\nu} + 2i\sigma_{\mu\nu} \tag{24a}$$

$$4_\mu \bar{\alpha}_\nu = \delta_{\mu\nu} + 2i\bar{\sigma}_{\mu\nu} \tag{24b}$$

where

$$\sigma_{i4} = -\bar{a}_{i4} = 1/2\ a, \tag{24c}$$

$$\sigma_{ij} = \bar{\sigma}_{..} = 1/2\ \varepsilon_{ijk}\sigma_k \tag{24d}$$

The antisymmetric matrices, $a_{\mu\nu}(\bar{\sigma}_{\mu\nu})$, are self (anti-self) dual, and satisfy the $O(4)$ algebra.

We rnultiply eq. (23a) by $\alpha_\nu(\partial_\nu - iA_\nu)$ and use (24a) to obtain

$$[(\partial_\mu - iA_\mu)^2 + \bar{\sigma}_{\mu\nu} F_{\mu\nu}]\ \chi_+ = 0 \tag{25}$$

For the gauge field (5a), we have

$$\bar{\sigma}_{\mu\nu} F_{\mu\nu} = (\vec{\sigma}.\hat{x})(\vec{T}.\hat{x}) \frac{(K^2(r) - K'(r)r-1)}{r^2} + (\vec{\sigma}.\vec{T}) \frac{K'(r)}{r^2}$$

$$- 2(\vec{\sigma}.\hat{x}) \; (\vec{T}.\hat{x})(\partial_t \partial_r \; W(r,t)) \tag{26}$$

We will use the regular monopole configuration, $K(r) = \%(r_M - r)$. For fermion energies $E \ll c$, the rnonopole core looks like point-like and we can make $r_M = 1/cg \to 0$. Then

$$\bar{\sigma}_{\mu\nu} F_{\mu\nu} = -\frac{1}{r^2} (\vec{\sigma}.\hat{x})(\vec{T}.\hat{x}) - 2(\vec{\sigma}.\hat{x})(\vec{T}.\hat{x})(\partial_r \partial_t W(r.t)) \tag{27}$$

For this gauge configuration with q=l (>0) we have $\partial_r \partial_t W(r,t) \geq 0$.

The operator $\vec{J} = \vec{L} + \vec{S} + \vec{T}$, the total angular moment, commutes with the Harniltonian defined in eqs. (20a-c). The zero mode wave function is characterized by $\vec{J}^2 = J_z = 0$. For this zero mode we have $(\vec{\sigma}.\hat{x})(\vec{T}.\hat{x}) < 0$, and then $\bar{\sigma}_{\mu\nu} F_{\mu\nu}$ is a non-negative operator, which is zero at $r=0$.

$(\partial_\mu - iA_\mu)^2$, defined over R^4 euclidean space, is a positive definite operator. Then

$$[(\partial_\mu - iA_\mu)^2 + \bar{\sigma}_{\mu\nu} F_{\mu\nu}] > 0$$

Consequently, by eq.(25), $\chi_+ = 0$. There are no normalizable zero modes with positive chirality for the $|M,1\rangle$ state. For the $|M,-1\rangle$ state, we have $\chi_- = 0$.

We conclude that for an arbitrary representation of isospin $\vec{T}$, and a monopole field $|M,r\rangle$, there will be

$$(\tfrac{2}{3})\; T(T+1)\; (2T+1) \; |r|$$

zero mode wave functions with definite chirality, as it happens for the vacuum sector[6].

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Resumo

São apresentadas certas relações de origem topológicas entre os instanton e os monopolos magnéticos. Em particular, mostramos os números de funções de onda de modos zero fermiônico com quiralidade definida são os mesmos para uma dada representação de Isospin do campo de gauge.