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New Algebraic Tables of SU(2) Quantitieo

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Abstract Formulas for Clebsch-Gordan Coefficients, 6-j and 9-j symbols of SU(2) are presented in a ready-to-program way for obtaining algebraic tables. An excerpt of the complete tables is also presented.

1. INTRODUCTION

Formulas for SU(2) Clebsch-Gordan Coefficients, 6-j and 9-j symbols can be found in any book on application of Group Theory to Quantum Mechanics[1-3]. These formulas can be easily programmed to produce *numerical* values. From them can also be obtained formulas with the simple structure of a polynomial multiplied by the square root of the ratio of two polynomials. We do that in the following sections.

Algebraic formulas of the SU(2) quantities with the structure mentioned above, besides having an interest by their own, are useful in several instances. With one single such algebraic formula one obtains not only one but an infinity of numerical values right away by using a pocket calculator (or even by hand calculation) just by ascribing numerical values to the variables. The formulas for the analogous quantities for higher dimensional unitary and orthogonal groups[4] have sometimes the SU(2) quantities as internal blocks. Hence, simpler algebraic formulas for the previous would require simpler algebraic formulas for the latter. These simpler algebraic formulas for higher dimensional unitary and orthogonal groups are also relevant in the development of the so called Microscopic Model of the Collective Nuclear Motion[5]. This

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was in fact our main motivation for the present work. Even the numerical evaluation of the SU(2) quantities will be benifited by these simpler algebraic formulas in the numerical precision as well in the computing time.

Another possible application of these formulas is the study of the non-trivial zeros of the Racah $(6\text{-}j)$ cuefficients[6].

The most extensive algebraic tables of these SU(2) quantities have been published by D.A. Varchalovitch, A. N. Moskalev and V.K. Khersonskii[7]. Our tables are an independent check[8] and an extension of their tables.

2. CLEBSCH-GORDAN COEFFICIENTS

For the SU(2) Clebsch-Gordan Coefficients we start from well known Racah's formula[1-3]

$$C^{j_1 j_2 j}_{m_1 m_2 m} = \delta(m_1+m_2, m)\left[\frac{(2j+1)(j_1+j_2-j)!(j_1-j_2+j)!(-j_1+j_2+j)!(j+m)!(j-m)!}{(j_1+j_2+j+1)!(j_1+m_1)!(j_1-m_1)!(j_2+m_2)!(j_2-m_2)!}\right]^{\frac{1}{2}}$$

$$\times \sum_z \frac{(-)^{z+j_2+m_2}(j+j_2+m_1-z)!(j_1-m_1+z)!}{z!(j+m-z)!(j-j_1+j_2-z)!(j_1-j_2-m+z)!} \qquad (2.1)$$

In order to obtain formulas with the structure mentioned in the introduction, like those given in Condon-Shortley's tables, one must sacrifice the generality of eq. (2.1) by assigning numerical values to some of its entries. The procedure is better seen if we re-write eq. (2.1) putting $j = j_1 + a$, $j_2 = b$; $m_2 = a$, thus obtaining

$$C^{j_1\ b\ j_1+a}_{m-\alpha\ \alpha\ m} =$$

$$= \left[\frac{(2j_1+2a+1)(b-a)!(2j_1+a-b)!(b+a)!(j_1+m+a)!(j_1-m+a)!}{(2j_1+b+a+1)!(j_1+m-\alpha)!(j_1-m+\alpha)!(b+\alpha)!(b-\alpha)!}\right]^{\frac{1}{2}}$$

$$\times \sum_z \frac{(-)^{b+\alpha+z}(j_1+m+a+b-\alpha-z)!(j_1-m+\alpha+z)!}{z!(j_1+m+a-z)!(a+b-z)!(j_1-m-b+z)!} \qquad (2.2)$$

To be able to perform the summation, the terms have to be expressed as ratios of polynomials. This is done by matching, for each factorial in numerator, a factorial in denominator such that the difference of their arguments be a numerical integer. From eq. (2.1) one sees that the only way to obtain this is by assigning numerical values to a, b and α.

By use of the following generalization of Pochhammer's symbol

$$
\begin{aligned}
P(f;k;\Delta) &= f(f+\Delta)(f+2\Delta)\ldots(f+(k-1)\Delta), \quad \text{for } k = \text{integer} > 0, \\
&= 1, \quad \text{for } k = 0,
\end{aligned}
\tag{2.3}
$$

and the relations

$$
b\pm\alpha = \text{integer} \gtrless 0 \;; \quad b\pm a = \text{integer} \gtrless 0 \;; \quad a+\alpha = \text{integer}, \tag{2.4}
$$

ensured by the branching rules, eq. (2.2) can be written as

$$
\begin{aligned}
C^{j_1\; b\; j_1+a}_{m-\alpha\; \alpha\; m} = {} & \left[\frac{(b-a)!\,(2j_1+2a+1)}{(b+\alpha)!\,(b-\alpha)!\,(b+a)!\,P(2j_1+a-b+1;2b+1;1)}\right]^{\frac{1}{2}} \\
& \times \left\{\begin{matrix} \left[2^{-(\mu-1)(a+\alpha)}P(\mu(j_1+m-\alpha+1);a+\alpha;\mu)\right]^{\frac{1}{2}} \\ \text{or} \\ \left[2^{(\mu-1)(a+\alpha)}P(\mu(j_1+m+a+1);-a-\alpha;\mu)\right]^{-\frac{1}{2}} \end{matrix}\right\} \\
& \times \left\{\begin{matrix} \left[2^{-(\mu-1)(a-\alpha)}P(\mu(j_1-m+\alpha+1);a-\alpha;\mu)\right]^{\frac{1}{2}} \\ \text{or} \\ \left[2^{(\mu-1)(a-\alpha)}P(\mu(j_1-m+a+1);-a+\alpha;\mu)\right]^{-\frac{1}{2}} \end{matrix}\right\} \\
& \times \frac{1}{2^{2(\mu-1)b}}\sum_{z=0}^{a+b}(-)^{b+\alpha+z}\binom{a+b}{z}P(\mu(j_1+m+a+1-z);b-\alpha;\mu) \\
& \qquad \times P(\mu(j_1-m-b+1+z);b+\alpha;\mu)
\end{aligned}
\tag{2.5}
$$

where

$$
\begin{aligned}
\mu &= 1\;, \quad \text{for even } b \\
\mu &= 2\;, \quad \text{for odd } b
\end{aligned}
$$

and in the round brackets one must take the terms that have P's with second argument positive.

All the numerical constants that appear in eq. (2.5) are integer, so it can be evaluated exactly using integer arithmetic. We have made a computer program to evaluate eq. (2.5) as an exact algebraic expression having j_1 and m as free variables and b , a and a as numerical values. This program produced a table of algebraic formulas for b =1/2, 1, 3/2,...,8 and $-b \leqslant \alpha$, $a \leqslant b$. In table I we present part of this table. [The complete table, as well those for 6-j and 9-j symbols, are available upon request from the author J.A.C.A].

3. (6-j) SYMBOLS

Taking the formula[1-3]

$$\begin{Bmatrix} j_1 j_2 j_3 \\ \ell_1 \ell_2 \ell_3 \end{Bmatrix} = (-1)^{j_1+j_2+\ell_1+\ell_2} \left[\frac{(-j_1+j_2+j_3)!(j_1+j_2-j_3)!(j_1-j_2+j_3)!(-j_1+\ell_2+\ell_3)!(j_1-\ell_2+\ell_3)!}{(j_1+j_2+j_3+1)!\ (j_1+\ell_2+\ell_3+1)!}\right.$$

$$\times \left.\frac{(j_1+\ell_2-\ell_3)!(-\ell_1+j_2+\ell_3)!(\ell_1-j_2+\ell_3)!(\ell_1+j_2-\ell_3)!(-\ell_1+\ell_2+j_3)!(\ell_1-\ell_2+j_3)!(\ell_1+\ell_2-j_3)!}{(\ell_1+j_2+\ell_3+1)!(\ell_1+\ell_2+j_3+1)!}\right]^{\frac{1}{2}}$$

$$\times \sum_z \frac{(-)^z\ (j_1+j_2+\ell_1+\ell_2+1-z)!}{z!(j_1+j_2-j_3-z)!(\ell_1+\ell_2-j_3-z)!(j_1+\ell_2-\ell_3-z)!(\ell_1+j_2-\ell_3-z)!(j_3+\ell_3-j_1-\ell_1+z)!(j_3+\ell_3-j_2-\ell_2+z)!} \tag{3.1}$$

and putting

$$j_1 = a\ ;\ \ j_2 = b\ ;\ \ j_3 = c\ ;\ \ell_2 = c+\alpha\ ;\ \ \ell_3 = b+\beta \tag{3.2}$$

one obtains

$$\begin{Bmatrix} a & b & c \\ \ell_1 & c+a & b+\beta \end{Bmatrix} = (-)^{a+b+c+\ell_1+\alpha} \left[\frac{(-a+b+c)!(a-b+c)!(a+b-c)!(-a+b+c+\alpha+\beta)!}{(a+b+c+1)!(a+b+c+\alpha+\beta+1)!}\right.$$

$$\times \left.\frac{(a+b-c-\alpha+\beta)!(a-b+c+\alpha-\beta)!(2b-\ell_1+\beta)!(\ell_1+\beta)!(\ell_1-\beta)!(2c-\ell_1+\alpha)!(\ell_1-\alpha)!(\ell_1+\alpha)!}{(2b+\ell_1+\beta+1)!(2c+\ell_1+\alpha+1)!}\right]^{\frac{1}{2}}$$

$$\times \sum_z \frac{(-)^z (a+b+c+\ell_1+\alpha+1-z)!}{z!(a+b-c-z)!(\ell_1+\alpha-z)!(a-b+c+\alpha-\beta-z)!(\ell_1-\beta-z)!(-a+b+c-\ell_1+\beta+z)!(-\alpha+\beta+z)!} \tag{3.3}$$

Table I - Algebraic expressions of Clebsch-Gordan Coefficients obtained from Eq.(2.5). For printing canvenience ue put J in place of j and M in place of m.

$$C^{J\;\;5/2\;\;J+5/2}_{M-1/2\;\;1/2\;\;M} = \left[\frac{5(2J+2M+1)(2J+2M+3)(2J+2M+5)(2J-2M+3)(2J-2M+5)}{16P(2J+1;5;1)}\right]^{1/2}$$

$$C^{J\;\;3\;\;J+1}_{M\;\;0\;\;M} = -\left[\frac{12(2J+3)(J+M+1)(J-M+1)}{P(2J-1;7;1)}\right]^{1/2}(-5M^2+J^2+2J)$$

$$C^{J\;\;7/2\;\;J+5/2}_{M-3/2\;\;3/2\;\;M} = -\left[\frac{3(2J+6)P(2J+2M-1;4;2)(2J-2M+5)}{128P(2J;8;1)}\right]^{1/2}(-14M+6J+21)$$

$$C^{J\;\;4\;\;J+3}_{M-2\;\;2\;\;M} = -\left[\frac{56(2J+7)P(J+M-1;5;1)(J-M+3)}{P(2J;9;1)}\right]^{1/2}(-2M+J+4)$$

$$C^{J\;\;4\;\;J+4}_{M\;\;0\;\;M} = \left[\frac{70P(J+M+1;4;1)P(J-M+1;4;1)}{P(2J+1;8;1)}\right]^{1/2}$$

$$C^{J\;\;5\;\;J+5}_{M-3\;\;3\;\;M} = \left[\frac{45P(J+M-2;8;1)(J-M+4)(J-M+5)}{P(2J+1;10;1)}\right]^{1/2}$$

$$C^{J\;\;5\;\;J+2}_{M-1\;\;1\;\;M} = \left[\frac{112(2J+5)(J+M)(J+M+1)(J+M+2)(J-M+2)}{P(2J-2;11;1)}\right]^{1/2}(15M^3-9JM^2-3J^2M-36M^2+J^3+6J^2+33M+5J-12)$$

$$C^{J\;\;11/2\;\;J+7/2}_{M-5/2\;\;5/2\;\;M} = \left[\frac{3(2J+8)P(2J+2M-3;6;2)(2J-2M+7)}{2048P(2J-1;12;1)}\right]^{1/2}(220M^2-200JM-1000M+28J^2+376J+1125)$$

$$C^{J\;\;6\;\;J+5}_{M-2\;\;2\;\;M} = -\left[\frac{660(2J+11)P(J+M-1;7;1)(J-M+3)(J-M+4)(J-M+5)}{P(2J;13;1)}\right]^{1/2}(-3M+J+6)$$

$$C^{J\;\;13/2\;\;J+13/2}_{M-11/2\;\;11/2\;\;M} = \left[\frac{13P(2J+2M-9;12;2)(2J-2M+13)}{8192P(2J+1;13;1)}\right]^{1/2}$$

$$C^{J\;\;13/2\;\;J+5/2}_{M-7/2\;\;7/2\;\;M} = \left[\frac{5(2J+6)P(2J+2M-5;6;2)(2J-2M+7)}{4096P(2J-3;14;1)}\right]^{1/2}(-40J^3-264J^2M+252J^2+1320JM^2-5544JM+5962J-1144M^3+8316M^2-20834M+18081)$$

$$C^{J\;\;7\;\;J+6}_{M-4\;\;4\;\;M} = -\left[\frac{104(2J+13)P(J+M-3;10;1)(J-M+5)(J-M+6)}{P(2J;15;1)}\right]^{1/2}(-7M+4J+28)$$

$$C^{J\;\;7\;\;J+6}_{M-2\;\;2\;\;M} = -\left[\frac{572(2J+13)P(J+M-1;8;1)P(J-M+3;4;1)}{P(2J;15;1)}\right]^{1/2}(-7M+2J+14)$$

As In the case of the Clebsch-Gordan coefficients, the generality must be sacrificed in order to have an expression with the desired simple structure. In this case the greatest generality is limited to 3 variables, We take a, b, c as free variables and ascribe numerical values to R, a and β. With this choice, there are in eq. (3.3) four numerical constraints on the values of z:

$$z \geqslant 0 \ ; \quad \ell_1+\alpha-z \geqslant 0 \ ; \quad \ell_1-\beta-z \geqslant 0; \quad -\alpha+\beta+z \geqslant 0 \tag{3.4}$$

from which it follows that

$$z_{min} = \max(0, \alpha-\beta) \ ; \qquad z_{max} = \min(\ell_1+\alpha, \ell_1-\beta) \tag{3.5}$$

There are then 4 possible ranges for z, according to the possibilities:

$$\begin{array}{ll} 1)\ \alpha+\beta \leqslant 0 \ \text{ and } \ \alpha-\beta \leqslant 0 \ ; & 2)\ a+\beta \leqslant 0 \text{ and } a-6 \geqslant 0 \\ 3)\ \alpha+\beta \geqslant 0 \ \text{ and } \ \alpha-\beta \leqslant 0 \ ; & 4)\ \alpha+\beta \geqslant 0 \text{ and } \alpha-\beta \geqslant 0 \end{array} \tag{3.6}$$

Taking into account eq. (3.6) one can re-express eq. (3.3) in terms of generalized Pochhammer's symbols as

$$\begin{Bmatrix} a & b & c \\ \ell_1 & c+\alpha & b+\beta \end{Bmatrix} = \frac{(-)^s}{(\ell_1+\alpha)!} \left[\frac{P(n_1;k_1;1)P(s+n_s;k_s;1)P(s-2a+n_a;k_a;1)}{P(n_2;k_2;1)P(2b-\ell_1+\beta+1;2\ell_1+1;1)} \right.$$

$$\times \left. \frac{P(s-2b+n_b;k_b;1)P(s-2c+n_c;k_c;1)}{(2c-\ell_1+\alpha+1;2\ell_1+1;1)} \right]^{\frac{1}{2}} \times \sum_{z=z_{min}}^{z_{max}} (-)^{\ell_1+\alpha+z} \binom{\ell_1+\alpha}{z} P(s+\bar{n}_s;\bar{k}_s-z;1)$$

$$\times\, P(s-2a+\bar{n}_a+z;\bar{k}_a-z;1)P(s-2b+\bar{n}_b-z;\bar{k}_b+z;1)P(s-2c+\bar{n}_c-z;\bar{k}_c+z;1)$$

$$\times\, P(\ell_1-\beta-z+1;z;1)P(-\alpha+\beta+z+1;\ell_1+\alpha-z;1) \tag{3.7}$$

where

$$\begin{array}{l} s = a+b+c \ ; \quad \bar{n}_a = -\ell_1+\beta+1; \quad \bar{n}_b = \alpha-\beta+1 \\ \bar{n}_c = 1 \ ; \ k_a = k_s \ ; \ k_c = k_1 = k_2 = k_b \ ; \ \bar{k}_c = \bar{k}_b \end{array} \tag{3.8}$$

and the other parameters are given in the table 2, according to the four possibilities of eq. (3.6).

Table 2 - Parameters of eq. (3.7) for the four possibilities given in eq. (3.6).

Case	n_s	k_s	n_a	n_b	k_b	n_c	n_1	n_2	$\bar{n}_s$	$\bar{k}_s$	$\bar{k}_b$
1	$\alpha+\beta+2$	$-\alpha-\beta$	$\alpha+\beta+1$	$\alpha-\beta+1$	$-\alpha+\beta$	1	$\ell_1-\beta+1$	$\ell_1+\alpha+1$	2	$\ell_1+\alpha$	0
2	$\alpha+\beta+2$	$-\alpha-\beta$	$\alpha+\beta+1$	1	$\alpha-\beta$	$-\alpha+\beta+1$	$\ell_1+\beta+1$	$\ell_1-\alpha+1$	2	$\ell_1+\alpha$	$-\alpha+\beta$
3	2	$\alpha+\beta$	1	$\alpha-\beta+1$	$-\alpha+\beta$	1	$\ell_1-\beta+1$	$\ell_1+\alpha+1$	$\alpha+\beta+2$	$\ell_1-\beta$	0
4	2	$\alpha+\beta$	1	1	$\alpha-\beta$	$-\alpha+\beta+1$	$\ell_1+\beta+1$	$\ell_1-\alpha+1$	$\alpha+\beta+2$	$\ell_1-\beta$	$-\alpha+\beta$

For tabulation purposes it is enough to consider one of the 4 cases of eq. (3.6), since the other cases can be obtained by using the symmetries of the 6-j symbols. For general use however it is better to make a single program that includes all the cases.

We have made also a computer program to evaluate eq.(3.7) as an exact algebraic expression having a, b, c as free variables and ℓ_1, α and β as numerical values. This program produced a table of algebraic expressions for $\ell_1 = 1/2, 1, 3/2, \ldots, 8$ and all allowed values of α and β such that $\alpha+\beta \leqslant 0$ and $\alpha-\beta \leqslant 0$ (case 1 of eq. (3.6)). In table 3 we reproduce part of this table.

4. (9-*j*) SYMBOLS

We start again from a well known formula[1-3]

$$\begin{Bmatrix} a & b & c \\ \ell_1 & \ell_2 & \ell_3 \\ k_1 & k_2 & k_3 \end{Bmatrix} = \sum_z (-)^{2z}(2z+1) \begin{Bmatrix} a & k & z \\ \ell_3 & b & c \end{Bmatrix} \begin{Bmatrix} k_2 & \ell_1 & z \\ a & k_3 & k_1 \end{Bmatrix} \begin{Bmatrix} \ell_3 & b & z \\ k_2 & \ell_1 & \ell_2 \end{Bmatrix} \qquad (4.1)$$

Taking ℓ_1, ℓ_2, ℓ_3 as numerical constants, the branching rules require that k_1, k_2, k_3 have the form

Table III-Algebraic expressions uf $(6\text{-}j)$ symbols obtained from Eq.(3.8) The quantity X is defined as $-a(a+1)+b(b+1)+c(c+1)$. For printing convenience we put capital letters A,B,C in place of lou rase letters a,b,c and $P(f;k)$ in place of $P(f;k;1)$.

$$\begin{Bmatrix} A & B & C \\ 5/2 & C-5/2 & B+1/2 \end{Bmatrix} = (-)^{S} \left[\frac{10(S)(S+1)(S-2A-1)(S-2A)(S-2B-2)(S-2B-1)(S-2B)}{P(2B-1;6)} \cdot \frac{(S-2C+1)(S-2C+2)(S-2C+3)}{P(2C-4;6)}\right]^{1/2}$$

$$\begin{Bmatrix} A & B & C \\ 3 & C-1 & B \end{Bmatrix} = (-)^{S} \left[\frac{12(S+1)(S-2A)(S-2B)(S-2C+1)}{P(2B-2;7)P(2C-3;7)}\right]^{1/2} (4(B^2+B+2C^2+3-B^2C^2-BC^2+5C)-10X(2+C)+5X^2)$$

$$\begin{Bmatrix} A & B & C \\ 3 & C-2 & B+1 \end{Bmatrix} = (-)^{S+1} \left[\frac{10(S+1)(S-2A)(S-2B-2)(S-2B-1)(S-2B)(S-2C+1)(S-2C+2)}{P(2B-1;7)} \cdot \frac{(S-2C+3)}{P(2C-4;7)}\right]^{1/2} (2(BC+B-2C-2)+3X)$$

$$\begin{Bmatrix} A & B & C \\ 4 & C-4 & B \end{Bmatrix} = (-)^{S} \left[\frac{70P(S-2;4)P(S-2A-3;4)P(S-2B-3;4)P(S-2C+1;4)}{P(2B-3;9)P(2C-7;9)}\right]^{1/2}$$

$$\begin{Bmatrix} A & B & C \\ 9/2 & C-7/2 & B+7/2 \end{Bmatrix} = (-)^{S+1} \left[\frac{P(S-2B-6;7)P(S-2C+1;7)}{P(2B;10)P(2C-7;10)}\right]^{1/2} (14(BC+B)+9X)$$

$$\begin{Bmatrix} A & B & C \\ 5 & C-5 & B+3 \end{Bmatrix} = (-)^{S} \left[\frac{45(S)(S+1)(S-2A-1)(S-2A)P(S-2B-7;8)P(S-2C+1;8)}{P(2B-1;11)P(2C-9;11)}\right]^{1/2}$$

$$\begin{Bmatrix} A & B & C \\ 11/2 & C-7/2 & B+5/2 \end{Bmatrix} = (-)^{S} \left[\frac{3(S+1)(S-2A)P(S-2B-5;6)P(S-2C+1;6)}{P(2B-2;12)P(2C-8;12)}\right]^{1/2} (4(7B^2C^2+45B^2C+38B^2-130BC-67B+56C^2+140C+84-63BC^2)+10X(10BC+15B-21C-37)+55X^2)$$

$$\begin{Bmatrix} A & B & C \\ 13/2 & C-13/2 & B+11/2 \end{Bmatrix} = (-)^{S} \left[\frac{13(S+1)(S-2A)P(S-2B-11;12)P(S-2C+1;12)}{P(2B;14)P(2C-12;14)}\right]^{1/2}$$

$$\begin{Bmatrix} A & B & C \\ 13/2 & C-7/2 & B+5/2 \end{Bmatrix} = (-)^{S+1} \left[\frac{10(S+1)(S-2A)P(S-2B-5;6)P(S-2C+1;6)}{P(2B-3;14)P(2C-9;14)}\right]^{1/2} \cdot$$

$$\cdot(-8(5B^3C^3-60B^3C^2-215B^3C-150B^3+84B^2C^3+455B^2C^2+920B^2C+549B^2-341BC^3-1375BC^2-1595BC-561B+252C^3+1134C^2+1638C+756)+44X(3B^2C^2+30B^2C+35B^2-42BC^2-135BC-115B+53C^2+200C+201)+22X^2(15BC+30B-42C-110)+143X^3)$$

$$k_1 = a + \alpha; \; k_2 = b + \beta; \; k_3 = c + \gamma \tag{4.2}$$

with

$$-\ell_1 \leqslant \alpha \leqslant \ell_1 \; ; \; -\ell_2 \leqslant \beta \leqslant \ell_2 \; ; \; -\ell_3 \leqslant \gamma \leqslant \ell_3 \tag{4.3}$$

Therefore eq. (4.1) can be re-written as

$$\begin{Bmatrix} a & b & c \\ \ell_1 & \ell_2 & \ell_3 \\ a+\alpha & b+\beta & c+\gamma \end{Bmatrix} (-)^{2b} \sum_{\delta} (-)^{2\delta}(2b+2\delta+1) \begin{Bmatrix} a & b & c \\ \ell_3 & c+\gamma & b+\delta \end{Bmatrix}$$

$$\times \begin{Bmatrix} (c+\gamma) & a & (b+\delta) \\ \ell_1 & (b+\delta)+(\beta-\delta) & a+\alpha \end{Bmatrix} \begin{Bmatrix} (b+\delta) & (\ell_1+\lambda) & b \\ \ell_2 & b+\beta & (\ell_1+\lambda)-\lambda \end{Bmatrix} \tag{4.4}$$

where $\lambda = \ell_3 - \ell_1$ and δ runs through $-\ell_3 \leqslant \delta \leqslant \ell_3$ subject to $-\ell_1 \leqslant (\beta-\delta) \leqslant \ell_1$.

The entanglement of the $(6\text{-}j)$ syrnbols in eq. (4.4) and the need of having ℓ_1, α and β in eq. (3.7) as numerical values prevent us of obtaining general tables of $(9\text{-}j)$ symbols in more than 3 free variables.

We have made a computer program to evaluate eq. (4.4) as an exact algebraic expression using a subroutine to evaluate the (6-j) symbols. This program produced a table of algebraic expressions for the $(9\text{-}j)$ symbols, eq. (4.4), with $\ell_1 \leqslant \ell_2 \leqslant \ell_3 \leqslant 3$, for $\ell_1 = 1/2, 1, 3/2, 2, 5/2, 3$ and all porssibillties for a, β and γ. Using the symmetries of the $(9\text{-}j)$ syrnbols, all the case with $\ell_i \leqslant 3$ are covered by this table. In table 4 we present some items of the complete table.

In despite of the apparent simplicity of eq. (4.4), the coding of a program to evaluate it as an exact algebraic expression reaches practically the greatest degree of cornplexity in group theoretical algebraic calculations.

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Tahle IV - Algebraic expressions of **(9-J)** symbols obtained from Eq.(4.4) **The** quantity Z is defined **as** $a(a+1)+b(b+1)-c(c+1)$. **For printing** conve- nience **we** put **capital** letters A,B,C in **place of low case** letters a,b,c **and** $P(f;k)$ in **place of** $P(f;k;1)$.

$$\begin{Bmatrix} A & B & C \\ 1 & 1 & 1 \\ A & B & C-1 \end{Bmatrix} = \left[\frac{4(S+1)(S-2A)(S-2B)(S-2C+1)}{3(2A)(2A+1)(2A+2)(2B)(2B+1)(2B+2)(2C-1)(2C)(2C+1)}\right]^{1/2} C$$

$$\begin{Bmatrix} A & B & C \\ 3/2 & 1/2 & 1 \\ A-1/2 & B-1/2 & C \end{Bmatrix} = \left[\frac{(S+1)(S-2C)}{6P(2A-1;4)(2B)(2B+1)(2C)(2C+1)(2C+2)}\right]^{1/2} (2(AB+B+2A^2+2A)-3Z)$$

$$\begin{Bmatrix} A & B & C \\ 3/2 & 2 & 5/2 \\ A-3/2 & B+1 & C+5/2 \end{Bmatrix} = \left[\frac{2(S+2)(S+3)P(S+1;5)(S-2C-2)(S-2C-1)(S-2C)}{7P(2A-2;4)P(2B;5)P(2C+1;6)}\right]^{1/2} (2C-B+2A+2)$$

$$\begin{Bmatrix} A & B & C \\ 2 & 2 & 3 \\ A+1 & B+2 & C-2 \end{Bmatrix} = \left[\frac{12(S+2)(S-2A)(S-2B-2)S-2B-1)(S-2B)(P(S-2C+1;5)}{7P(2A;5)P(2B+1;5)P(2C-4;7)}\right]^{1/2} \cdot (2BC-AC+2C+2B^2-AB+4B-A^2-3A+2)$$

$$\begin{Bmatrix} A & B & C \\ 5/2 & 5/2 & 3 \\ A+5/2 & B+5/2 & C+3 \end{Bmatrix} = \left[\frac{25P(S+2;8)(S-2A+1)(S-2A+2)(S-2A+3)(S-2B+1)(S-2B+2)}{63P(2A+1;6)P(2B+1;6)} \cdot \frac{(S-2B+3)(S-2C+1)(S-2C+2)}{P(2C+1;7)}\right]^{1/2}$$

$$\begin{Bmatrix} A & B & C \\ 5/2 & 5/2 & 3 \\ A+3/2 & B+3/2 & C-2 \end{Bmatrix} = -\left[\frac{2(S+2)(S-2A-1)(S-2A)(S-2B-1)(S-2B)P(S-2C+1;5)}{21P(2A;6)P(2B;6)P(2C-4;7)}\right]^{1/2} \cdot (2(-10B^2C-10BC+10A^2C+10AC-10B^3+AB^2-20B^2-A^2B-10B+10A^3+20A^2+10A)+15Z(B-A))$$

$$\begin{Bmatrix} A & B & C \\ 5/2 & 5/2 & 3 \\ A+1/2 & B+5/2 & C+1 \end{Bmatrix} = \left[\frac{2P(S+2;4)(S-2A+1)(S-2A+2)(S-2A+3)(S-2B)(S-2C+1)}{21P(2A-1;6)P(2B+1;6)} \cdot \frac{(S-2C+2)}{P(2C-1;7)}\right]^{1/2} ((20B^2C-10ABC+80BC-10A^2C-31AC+78C-20B^3+14AB^2-92B^2+20A^2B+69AB-132B-6A^3+7A^2+58A-60)+Z(5C-5B+A+2))$$

$$\begin{Bmatrix} A & B & C \\ 3 & 3 & 3 \\ A+1 & B+2 & C+3 \end{Bmatrix} = -\left[\frac{4P(S+2;6)P(S-2A+1;4)(S-2B+1)(S-2B+2)}{21P(2A-1;7)P(2B;7)P(2C+1;7)}\right]^{1/2} (2(-15B^2C-10ABC+5BC-6A^2C+3AC+10AB^2-5B^2+14A^2B+8AB+6A^3-3A^2+15B^3)+15Z(C-2B-A+1))$$

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8. We detected two errors in the 9-j tables of Ref.7. The correct expressions are given by the first two entries of our table 4.

Resumo

Apresentam-se fórmulas para Coeficientes de Clebsch - Gordan, símbolos 6-j e símbolos 9-j apropriadas para construção de tabelas algébricas via computador. Um resumo das tabelas completas obtidas dessas fórmulas é também apresentado.