Revista Brasileira de Física, Vol. 14, nº 1, 1984

Extension of the Decomposition Theorem for Ising and Classical Spin Systems

L. L. **GONÇALVES**

Departamento de Física, Universidade Federal do Ceará, Caixa Postal 3004, Fortaleza, 60000, Ceará, Brasil

Recebido em 27 de março de 1984

Abstract It is considered a system composed of two subsystems consisting of Ising and classical v-dimensional spins, which interact through multiple decorated bonds. It is proved that under suitable conditions the canonical average of an arbitrary function of the spins in one of the subsystems does not depend on the interaction parameters contained in the other.

We have recently considered[1] two mixed spin systems in d-dimensional lattices consisting of classical and Ising spin systems. The classical spin $\vec{S}_0$ is a unit vector with ν components, and $\sigma_\ell = \pm 1$, where the label R identifies the lattice site. The Hamiltonian of the system A, $H_A(\sigma_1^A,\ldots\sigma_N^A;\vec{S}_1^A,\ldots\vec{S}_N^A,)$, was considered to be of the most general form and the Hamiltonian of the system B, $H_B(\sigma_1^B,\ldots\sigma_M^B;\vec{S}_1^B,\ldots,\vec{S}_M^B,)$, was restricted to a class of systems where the Ising spins appear in even products or where the classical spins generate these interactions. We also considered that the two systems interact through a single decorated bond via the interaction Hamiltonian

$$H_{AB} = -J_{jk}(\sigma_j^A + \sigma_k^B) \cdot S_{jk}^\nu \qquad (1)$$

where S_{jk}^ν is the vth component of the classical spin $\vec{S}_{jk}$ on the bond connecting site j on system A to site k on system B. We have proved under the above assumptions[1] the following theorem: "The canonical ensemble average of any function of the Ising and/or classical spins belonging to system A does not depend on the interaction parameters contained in H_{AB} and H_B". This result is an extension of the result obtained by Falk[2] for systems containing Ising spins only. Naturally in this case the classical spin in the interaction Hamiltonian, eq. (1), is substituted by an Ising spin.

A preliminary version of this paper has been presented at the 15th International Conference on Thermodinamics and Statistical Mechanics, Edinburgh, 1983.

Work partially financed by the Brazilian Agencies CNPq and FINEP.

Let us consider more complex systems A and B whose Hamiltonians $H_A(\sigma_1^A,..\sigma_N^A;\bar{\sigma}_1^A,..,\bar{\sigma}_{N'}^A;\vec{S}_1^A,..,\vec{S}_{N''}^A)$ and $H_B(\sigma_1^B,..,\sigma_M^B;\bar{\sigma}_1^B,..,\bar{\sigma}_{M'}^B;\vec{S}_1^B,..,\vec{S}_{M''}^B)$ depend also on Ising spins $\bar{\sigma}_\ell$ of magnitude s. The Hamiltonian H_A is considered to be of the most general form. This means that it can contain all possible interaction terms formed with the set of spins $\{\sigma_1^A,...,\sigma_N^A;\bar{\sigma}_1^A,...,\bar{\sigma}_{N'}^A;\vec{S}_1^A,...\vec{S}_{N''}^A\}$. The Hamiltonian H_B is restricted to the same conditions imposed on reference 1 and described above. Explicity it can contain for instance terms of the form $J_{\ell_1\ell_2}\sigma_{\ell_1}^B\sigma_{\ell_2}^B$, $J_{\ell_1\ell_2\ell_3\ell_4}\sigma_{\ell_1}^B\sigma_{\ell_2}^B\sigma_{\ell_3}^B\sigma_{\ell_4}^B\ldots$ etc, $J_{\ell_1\ell_2\ell_3}\bar{\sigma}_{\ell_1}(\sigma_{\ell_2}^B+\sigma_{\ell_3}^B)$, $\tilde{J}_{\ell_1\ell_2\ell_3}S_{\ell_1}^{B\nu}(\sigma_{\ell_2}^B+\sigma_{\ell_3}^B)$ for arbitrary $\ell_1,\ell_2,\ell_3,\ell_4$, and $\bar{\bar{J}}_{\ell_1'\ell_2'}\bar{\sigma}_{\ell_1'}^B\bar{\sigma}_{\ell_2'}^B$, $\tilde{\tilde{J}}_{\ell_1'\ell_2'}S_{\ell_1'}^{B\nu}S_{\ell_2'}^{B\nu}$ for ℓ_1', ℓ_2' from ℓ_1 and ℓ_2, where $S_\ell^{B\nu}$ is νth component of the classical rpin $\vec{S}_\ell^B$.

The purpose of this note is to prove that the abovetheoremis still valid when we allow multiple interactions between the system A and B through decorated bonds, and the Interaction Hamiltonian is written in the form:

$$H_{AB} = -\sum_{n=1}^{m} J_{j,k+n}\,\sigma_j^A\sigma_{k+n}^B - \sum_{n=1}^{m}\tilde{J}_{j,k+n}\,S_{j,k+n}^{\nu}(\sigma_j^A+\sigma_{k+n}^B)$$
$$-\sum_{n=1}^{m}\bar{J}_{j,k+n}\,\bar{\sigma}_{j,k+n}(\sigma_j^A+\sigma_{k+n}^B) \qquad (2)$$

Let

$$f(\sigma^A,\bar{\sigma}^A,\vec{S}^A) \equiv f(\sigma_1^A,...,\sigma_N^A;\bar{\sigma}_1^A,...,\bar{\sigma}_{N'}^A;\vec{S}_1^A,...,\vec{S}_{N''}^A)$$

be an arbitrary function of the classical and Ising spins of the system A. Then we have

$$\langle f(\sigma^A,\bar{\sigma}^A,\vec{S}^A)\rangle = \frac{\displaystyle\sum_{\{\sigma^A,\bar{\sigma}^A,\sigma^B,\bar{\sigma}^B,\bar{\sigma}_{j,k+n}\}}\int\prod_{i=1}^{N''}d\Omega_i^A\prod_{i=1}^{M''}d\Omega_i^B\prod_{n=1}^{m}d\Omega_{j,k+n}\,f(\sigma^A,\bar{\sigma}^A,\vec{S}^A)\exp[-\beta(H_A+H_B+H_{AB})]}{\displaystyle\sum_{\{\sigma^A,\bar{\sigma}^A,\sigma^B,\bar{\sigma}^B,\bar{\sigma}_{j,k+n}\}}\int\prod_{i=1}^{N''}d\Omega_i^A\prod_{i=1}^{M''}d\Omega_i^B\prod_{n=1}^{m}d\Omega_{j,k+n}\exp[-\beta(H_A+H_B+H_{AB})]} \qquad (3)$$

where $d\Omega_i^A$, $d\Omega_i^B$ and $d\Omega_{j,k+n}$ are the elements of the hypersolid anglefor the vectors $\vec{S}_i^A$, $\vec{S}_i^B$ and $\vec{S}_{j,k+n}$ respectively.

Since we know H_{AB}, the integrals on the hypersolid angle $d\Omega_{j,k+n}$ can be performed immediately and we get the result[1]

$$<f(\sigma^A,\bar{\sigma}^A,\vec{S}^A)> = \frac{\displaystyle\sum_{\{\sigma^A,\bar{\sigma}^A,\sigma^B,\bar{\sigma}^B,\bar{\sigma}_{j,k+n}\}} \int \prod_{i=1}^{N''} d\Omega_i^A \prod_{i=1}^{M''} d\Omega_i^B f(\sigma^A\bar{\sigma}^A,\vec{S}^A)\exp(-\beta H_B)\exp(-\beta H'_{AB})}{\displaystyle\sum_{\{\sigma^A,\bar{\sigma}^A,\sigma^B,\sigma^B,\bar{\sigma}_{j,k+n}\}} \int \prod_{i=1}^{N''} d\Omega_i \prod_{i=1}^{M''} d\Omega_i^B \exp(-\beta H_A)\exp(-\beta H_B)\exp(-\beta H'_{AB})} \tag{4}$$

where

$$H'_{AB} = -\sum_{n=1}^{m} (J_{j,k+n} + \tilde{J}'_{j,k+n})\sigma_j^A\sigma_{k+n}^B - \sum_{n=1}^{m} \bar{J}_{j,k+n}\,\bar{\sigma}_{j,k+n}(\sigma_j^A + \sigma_{k+n}^B) \tag{5}$$

with

$$\tilde{J}'_{j,k+n} = \frac{1}{2} K_B T \,\ell n\left[\Gamma(\nu/2)\,(\beta\tilde{J}_{j,k+n})^{1-\nu/2}\, I_{\nu/2-1}\,(2\beta\tilde{J}_{j,k+n})\right] \tag{6}$$

and $I_{\nu/2-1}$ is the modified Bessel function.

Evaluating the trace on the spins $\bar{\sigma}_{j,k+n}$ we get imnediately[3]

$$<f(\sigma^A,\bar{\sigma}^A,\vec{S}^A)> = \frac{\displaystyle\sum_{\{\sigma^A,\bar{\sigma}^A,\sigma^B,\bar{\sigma}^B\}} \int \prod_{i=1}^{N''} d\Omega_i^A \prod_{i=1}^{M''} d\Omega_i^B\, f(\sigma^A,\bar{\sigma}^A,\vec{S}^A)\exp(-\beta H_A)\exp(-\beta H_B)\exp(-\beta H''_{AB})}{\displaystyle\sum_{\{\sigma^A,\bar{\sigma}^A,\sigma^B,\bar{\sigma}^B\}} \int \prod_{i=1}^{N''} d\Omega_i^A \prod_{i=1}^{M''} d\Omega_i^B \exp(-\beta H_A)\exp(-\beta H_B)\exp(-\beta H''_{AB})} \tag{7}$$

where

$$H''_{AB} = -\sum_{n=1}^{m} (J_{j,k+n} + \tilde{J}'_{j,k+n} + \tilde{J}'_{j,k+n})\sigma_j^A\sigma_{k+n}^B \tag{8}$$

and

$$\bar{J}'_{j,k+n} = \frac{K_B T}{2} \quad \ell n \left[\frac{\sinh[\beta\bar{J}(2s+1)]}{(2s+1)\sinh(\beta\bar{J})}\right] \tag{9}$$

Introducing[2] the new spin variable $t_R = \pm 1$ such that

$$\begin{aligned} \sigma^A_\ell &= t^A_\ell \\ \sigma^B_\ell &= t^A_j t^B_\ell \end{aligned} \tag{10}$$

we can write H_A, H_B and H_{AB} in the form

$$\begin{aligned} \bar{H}_A &\equiv H_A(t^A_1,\ldots,t^A_N;\bar{\sigma}^A_1,\ldots,\bar{\sigma}^A_{N'};\vec{S}_1,\ldots,\vec{S}^A_{N''}) \\ \bar{H}_B &\equiv H_B(t^B_{11},\ldots,t^B_{MM};\bar{\sigma}^B,\ldots,\bar{\sigma}^B_{M'};\vec{S}^B_1,\ldots,\vec{S}^B_{M''}) \\ \bar{H}''_{AB} &\equiv H''_{AB}\ (t^B_{k+1},\ldots,t^B_{k+m}) \end{aligned} \tag{11}$$

where

$$t^B_{\ell k} \equiv t^B_\ell t^B_k = \sigma^B_\ell \sigma^B_k$$

Substituting these results in eq.(7) we obtain

$$\langle f(\sigma^A,\bar{\sigma}^A,\vec{S}^A)\rangle = \frac{\sum\limits_{\{t^A,\bar{\sigma}^A\}} \int \prod\limits_{i=1}^{N''} d\Omega^A_i\ (-\beta\bar{H}_A) f \quad \sum\limits_{\{t^B,\bar{\sigma}^B\}} \int \prod\limits_{i=1}^{M''} d\Omega^B_i\ \exp[-\beta(\bar{H}_B + H''_{AB})]}{\sum\limits_{\{t^A,\bar{\sigma}^A\}} \int \prod\limits_{i=1}^{N''} d\Omega^A_i\ (-\beta\bar{H}_A) \quad \sum\limits_{\{t^B,\bar{\sigma}^B\}} \int \prod\limits_{i=1}^{M''} d\Omega^B_i\ \exp[-\beta(\bar{H}_B + H''_{AB})]} \tag{12}$$

From the previous equation there follows immediately the result

$$\langle f(\sigma^A,\bar{\sigma}^A,\vec{S}^A)\rangle = \frac{\sum\limits_{\{t^A,\bar{\sigma}^A\}} \int \prod\limits_{i=1}^{N''} d\Omega^A_i\ \exp(-\beta\bar{H}_A)\ f}{\sum\limits_{\{t^A,\bar{\sigma}^A\}} \int \prod\limits_{i=1}^{N''} d\Omega^A_i\ \exp(-\beta\bar{H}_A)} \tag{13}$$

which proves our initial statement.

One immediate application of this result is the calculation of the two-spin correlation function on a Cayley tree where we have alternate next-nearest-neighbour interaction. By alternate next-nearest-neighbour interaction we mean that the bonds connecting the next-nearest-neighbours do not cross. By using the result proved above it is possible after a pruning of the lattice to end up with a chain composed of triangles connected by sides and vertices.

REFERENCES

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Resumo

É considerado um sistema composto de dois sub-sistemas constituídos de spins de Ising e spins clássicos de dimensão V, que interagern através de múltiplas ligações decoradas. É mostrado que sob condições adequadas a média canônica de uma função arbitrária dos spins em um dos sub-sistemas não depende dos parâmetros de interaçao contidos no outro.