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A Variational Calculation for the Charmonium

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Abstract In this paper we consider the non-relativistic approximation for the $c\bar{c}$ system in order to study the J/ψ family spectrum and some other properties of these resonances such as hadronic, leptonic, and electric dipole transition widths. The potential used is phenomenological, with a spin dependent part, of the types square root and linear. The Schrödinger equation is solved with the variational method where the configuration space is spanned by a set of harmonic oscillator wave functions.

1. INTRODUCTION

The spectrum and decay properties of the J/ψ family have been the subject of a number of studies in recent years. These resonances can be inkerpreted as bound states of the charmed quark-antiquark ($c\bar{c}$) system. Experimental efforts to find free quarks have been unsuccessful and therefore it is made the assumption that quarks are confined somehow. The charmed quarks are supposed to be so heavy that a non-relativistic treatment is justified.

In this paper we investigate the applicability ofavariational method — where configuration space is spanned by a set of harmonic oscillator (HO) wave functions — in order to study the mass spectrum and some decay properties (hadronic, leptonic, and electric dipole transition widths) of the charmonium. The confinement is achieved through a phenomenological potential of the types square root and linear. Its parameters are chosen by fitting the \$' - \$ mass difference and the \$ mass.

2. CALCULATION

We start with the intrinsic hamiltonian for two particles of masses m_1 and m_2, interacting via the potential $V(r')$

$$H_I = \frac{p'^2}{2\mu} + V(r') \qquad (1)$$

where $\vec{r}' = \vec{r}_1' - \vec{r}_2'$ is the relative coordinate, $\mu = m_1 m_2/(m_1+m_2)$ is the reduced mass, and $\vec{p}' = -i\hbar\vec{\nabla}'$. Primed variables means they have dimension.

Considering, on the other hand, the same two particles interacting now through the harmonic oscillator potential $\frac{1}{2}\mu\omega^2 r'^2$ the corresponding hamiltonian is given by

$$H_{HO} = \frac{p'^2}{2\mu} + \frac{1}{2}\mu\omega^2 r'^2 \tag{2}$$

Therefore

$$H_I = H_{HO} - \frac{1}{2}\mu\omega^2 r'^2 + V(r') \tag{3}$$

lntroducing dimensionless (unprimed) coordinate and momentum

$$\vec{r}' = \sqrt{\frac{\hbar}{\mu\omega}}\,\vec{r} \tag{4}$$

$$\vec{p}' = \sqrt{\mu\hbar\omega}\,\vec{p} \tag{5}$$

the above hamiltonians become

$$H_I = H_{HO} - \frac{\varepsilon}{2} r^2 + V(r) \tag{6}$$

$$H_{HO} = \frac{\varepsilon}{2}(p^2 + r^2) \tag{7}$$

where

$$\varepsilon = \hbar\omega \tag{8}$$

is taken as a non-linear variational parameter.

Each eigenfunction of H_I is expanded in a basis of eigenfunctions of H_{HO} with well defined spin, orbital and total angular momenta, i.e., the trial wave function is

$$\Psi = \sum_{\nu} a_\nu \,|\nu\, S,\, JM\rangle \tag{9}$$

with

$$|\nu\, S;\, JM\rangle = \sum_{m} (\ell S m,\, M-m|JM)\;\; |n\ell m\rangle |S, M-m\rangle \tag{10}$$

and ν stands for the pair of HO quantum numbers $(n\ell)$; a_ν are taken as linear variational parameters.

In order to obtain the mass spectrum we diagonalize the matrix $||H_I||$ whose elements are calculated with respect to the wave functions

given in eq.(10). In the diagonalization process we get the eigenvalues (energy or mass spectrum) and the eigenfunctionsof eq.(6) (i.e., the coefficients a_ν). We considered a basis of dimension 28 in our calculations, that is, $0 \leq n \leq 27$ in the expansion (9). We verify thatthis dimension is sufficient to assure the convergence in that expansion by diagonalizing submatrices of $||H_I||$. For instance, comparing the masses obtained using a basis of dimension 20 with the ones calculated withthe dimension 28 we notice that there is a variation of no more than 0.1%. It is chosen a value of ε which minimizes the lowest eigenvalue foreach set of (ℓSJ) states.

Following ref. 1, we assume that the $c\bar{c}$ potential is

$$V(r') = V_0(r') + V_{SS}(r') + V_{\ell S}(r') \tag{11}$$

where the confining and coulombic parts are given by

$$V_0(r') = Ar'^{\gamma} - \frac{B}{r'} + C \tag{12}$$

with $B = \frac{4}{3} a_s \hbar c$, and the spin dependent parts which come from a generalization of the Fermi-Breit potential[2] are

$$V_{SS}(r') = \frac{2}{3m_c^2 c^2} \vec{S}_1 . \vec{S}_2 \quad \nabla'^2 V_0(r') \tag{13}$$

$$V_{\ell S}(r') = \frac{3}{2m_c^2 c^2} \vec{\ell} . \vec{S} \frac{1}{r'} \frac{dV_0}{dr'} \tag{14}$$

where $\vec{S} = \vec{S}_1 + \vec{S}_2$ and m_c is the mass of quark c. Here we consider only the cases $\gamma = 1/2$ (square root potential) and $\gamma = 1$ (linear potential). As mentioned earlier, the parameters A and C are chosen so that to fit respectively the $\psi' - \psi$ mass difference and the ψ mass. The strong coupling constant a_s was varied from 0 to 0.2.

In comparison with the numerical calculation of ref. 1 ourforlism has the advantage of including a priori the spln dependent parts (13) and (14), which means that the wave functions obtained here are expected to describe fully each state caracterized by the quantum numbers (ℓSJ).

The matrix elements of $||H_I||$ using the above potential can be found to be

$$\langle n'\ell SJ|H_I|n\ell SJ\rangle = \frac{\varepsilon}{2}\{\sqrt{n(n+\ell+\frac{1}{2})}\,\delta_{n',n-1} + (2n+\ell+\frac{3}{2})\,\delta_{n'n} + \sqrt{(n+1)(n+\ell+\frac{3}{2})}\,\delta_{n',n+1}\} + \sum_{p=\ell}^{n'+n+\ell} B(n'\ell,n\ell,p)\,\frac{1}{\Gamma(p+\frac{3}{2})}$$

$$\{A\,\xi^{\gamma}\,\Gamma(p+\frac{\gamma}{2}+\frac{3}{2}) - \frac{Bp!}{\xi} + A\gamma t\xi^{\gamma-2}\,\Gamma(p+\frac{\gamma}{2}+\frac{1}{2})\left[\frac{1}{3}(\gamma+1)\Delta_S+\frac{3}{4}\,\Delta_{J\ell S}\right] + \frac{3Bt\Delta_{J\ell S}}{4\xi^3}\,(p-1)!\,\} + \delta_{\ell 0}\,\frac{8Bt\Delta_S}{3\pi\,\xi^3}\,\sqrt{\frac{\Gamma(n+\frac{3}{2})\,\Gamma(n'+\frac{3}{2})}{n!\;n'!}} + C\delta_{n'n} \tag{15}$$

with

$$\xi = \frac{\hbar}{\sqrt{\mu\varepsilon}} \tag{16}$$

$$t = \left(\frac{\hbar}{m_e c}\right)^2 \tag{17}$$

$$\Delta_S = S(S+1) - \frac{3}{2} \tag{18}$$

$$\Delta_{J\ell S} = J(J+1) - \ell(\ell+1) - S(S+1) \tag{19}$$

and the coefficients $B(n'\ell, n\ell, p)$ are defined in ref. 3. For numerical computation reasons we used, instead of the above summation in p, the following expression[3]

$$\sum_p B(n'\ell,n\ell,p)\,f(p) = \frac{1}{2}\sum_{k=0}^{n}\sum_{k'=0}^{n'} a_{n\ell k}\,a_{n'\ell k'}\,\Gamma(p+\frac{3}{2})\,f(p) \tag{20}$$

where $p = k + k' + R$ and

$$a_{n\ell k} = \frac{(-1)^k\sqrt{2(n!)\,\Gamma(n+\ell+\frac{3}{2})}}{k!(n-k)!\;\Gamma(k+\ell+\frac{3}{2})} \tag{21}$$

Once the wave functions have been obtained according to the mentioned procedure we calculated some decay properties of the J/ψ family, namely, leptonic, hadronic, and electric dipole transition widths. The corresponding expressions are

leptonic decay width[4,5] (Van Royen-Weisskopf formula)

$$\Gamma(\psi \to e^+e^-) = \frac{16\pi\,\alpha^2\,e_c^2\,(\hbar c)^3}{} \left[1 - \frac{8\alpha_s(c\bar{c})}{3\pi}\right]^2 |\Psi(0)|^2 \qquad (22)$$

where $e_c = \frac{2}{3}$ is the charge of the quark c (in units of the proton charge) and α is the fine structure constant.

hadronic decay width[6]

$$\Gamma(\psi \to \text{hadrons}) = \frac{40(\pi^2-9)\,\alpha_s^3\,(\hbar c)^3}{81\,(m_c c^2)^2} |\Psi(0)|^2 \qquad (23)$$

In the expressions above the value of the squared wavefunction at the origin can be wrltten as (for S states)

$$|\Psi(0)|^2 = \langle \delta(\vec{r}\,') \rangle \approx \frac{2}{\pi^2\xi^3} \sum_{nn'} a_n a_{n'} \sqrt{\frac{\Gamma(n+\frac{3}{2})\Gamma(n'+\frac{3}{2})}{n!n'!}} \qquad (24)$$

electric dipole transition width[7]

$$\Gamma(E1) = \frac{4\alpha e_c^2}{27(\hbar c)^2} (2J_f+1) \left| \langle f || r' || i \rangle \right|^2 \Omega^3 \qquad (25)$$

where i and f stand for initial and final states, respectively, and R is their energy gap (we used calculated values). The above transition matrix element is

$$\langle f || r' || i \rangle = \left(\frac{2}{\frac{1}{\xi_i^2} + \frac{1}{\xi_f^2}} \right)^{1/2} \sum_{nn'} a_n a'_{n'} \sum_p C(n\ell, n'\ell', p, x) \frac{(p+1)!}{\Gamma(p+\frac{3}{2})} \qquad (26)$$

with $x = \xi_i/\xi_f$ and $(\ell+\ell')/2 \leq p \leq (\ell+\ell')/2 + n + n'$. The coefficients C are definad in ref. 8. When $\varepsilon_i = \varepsilon_f$ (i.e., $\xi_i = \xi_f = \xi$) it is preferable to use

$$\langle f || r' || i \rangle = \frac{\xi}{2} \sum_{nn'} a_n a'_{n'} \sum_{k=0}^{n} \sum_{k'=0}^{n'} a_{n\ell k}\, a_{n'\ell' k'}\, \Gamma(p+2) \qquad (27)$$

with $p = (\ell+\ell')/2 + k + k'$.

3. FIT AND RESULTS

We have performed the calculations with four different values for the charmed quark mass, viz., m_c = 1.5, 1.6, 1.8, 2.1 GeV/c^2. Using the last value we obtained a better overall agreement with the experimental J/ψ family mass spectrum. The results are presented in Figure 1 and Table 1, including the potential parameters and the values of E. We notice that for $\gamma = \frac{1}{2}$ and $\gamma \approx 1$, while a_s increases from 0 to 0.2, the agreement with the experimental masses[9] of the 1^3P_0, 1^3P_1 and 1^3P_2 states is improved. The results for the D states are low, in general.

In Table 2 are given the experimental[9] and calculated values for the leptonic, hadronic and E1 transition widths. The leptonic widths are overestimated a little. However the correction factors to the Van Royen-Weisskopf formula are not quite certain[5,10] and hence it is more convenient to compare the width ratios. For $\Gamma(\psi' \to e^+e^-)/\Gamma(\psi \to e^+e^-)$ we have 0.49, 0.58, and 0.45 in the three columns of Table 2, which are in reasonable agreement with the measured value of 0.46 ± 0.09. The hadronic widths agree well with the experimental values except in the case $\gamma = 1/2$, $a_s = 0.1$. This is due to their great sensibility on a_s, which appears raised to the third power in eq. (23). The electric dipole transition widths are overestimated though not so much as in the similar numerical calculation of ref. 1. The cases where experimental results are not known appear in more or less the same order of other calculations[1,7,11,12].

Analysing our results we notice that the square root potential with a_s = 0.2 is more adequated to describe the charmonium family than the linear potential. An improvement in the results could probably be achieved through the introduction of a tensor part in our potential as considered in ref. 2 and a calculation in this line is in progress. Finally, we conclude that this rather simple approach is a useful alternative tool to the study of the charmonium.

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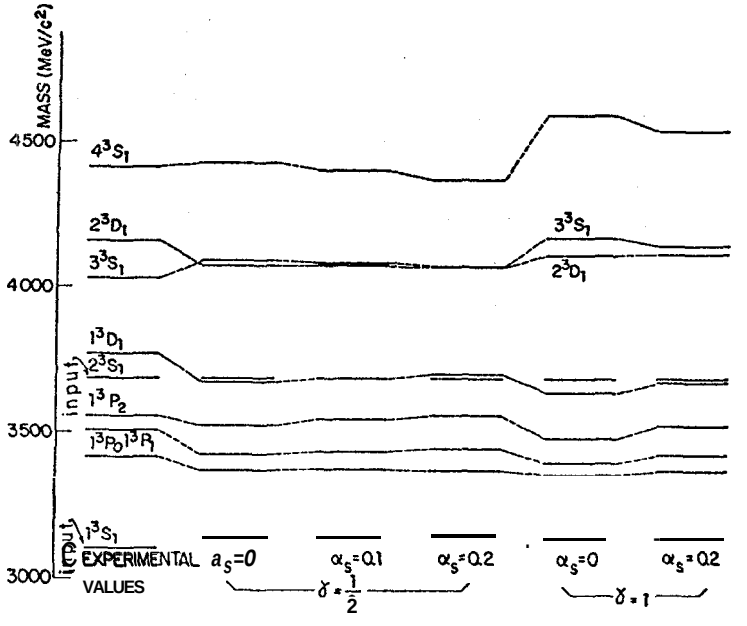

Fig. 1 - Calculated and experimental mass spectrum of the J/ψ family. The corresponding values of α_s and γ are indicated. m_c = 2.1 GeV/c^2.

Table 1 - Calculated and experimental values for the mass spectrum of the J/ψ family (in MeV/c^2). It is indicated all the potential parameters and also the values of E (in between parentheses, in MeV). m_c = 2.1 GeV/c^2.

STATE	EXPERIMENTAL VALUES	A=2148MeV.fm$^{-1/2}$ C=-2643.3 MeV α_s= 0; $\gamma = \frac{1}{2}$	A=1978MeV.fm$^{-1/2}$ C=-2450.4 MeV α_s = 0.1; $\gamma = \frac{1}{2}$	A=1795MeV.fm$^{-1/2}$ C=-2245.4 MeV α_s=0.2; $\gamma = \frac{1}{2}$	A=1450MeV.fm^{-1} C=-1910.9 MeV α_s= 0; γ = 1	A=1226MeV.fm^{-1} C=-1642.1 MeV α_s = 0.2; γ = 1
1^3S_1	3097.0±1.0 (Input)	3097(125)	3097(125)	3097(125)	3097(160)	3097(175)
2^3S_1	3685.3±2.7 (Input)	3684(125)	3685(125)	3685(125)	3686(160)	3685(175)
3^3S_1	4028.6±2.7	4096(125)	4086(125)	4072(125)	4170(160)	4143(175)
4^3S_1	4415.0±6.2	4427(125)	4404(125)	4374(125)	4598(160)	4540(175)
1^3P_0	3413.8±4.6	3366(125)	3369(125)	3369(125)	3353(195)	3362(195)
1^3P_1	3506.7±4.1	3421(125)	3432(125)	3442(125)	3396(175)	3420(165)
1^3P_2	3551.0±5.1	3520(125)	3539(125)	3556(125)	3480(190)	3520(190)
1^3D_1	3768.1±3.6	3671(130)	3686(125)	3700(115)	3640(145)	3673(155)
2^3D_1	4159.0±20.0	4076(130)	4074(125)	4069(115)	4122(145)	4116(155)

Table 2 - Calculated and experimental values for the leptonic, hadronic, and electric dipole transition widths (in keV).

TRANSITION	EXPERIMENTAL VALUES	$\gamma = \frac{1}{2}$		$\gamma = 1$
		$\alpha_s = 0.1$	$\alpha_s = 0.2$	$\alpha_s = 0.2$
$\Gamma(\psi \to e^+e^-)$	4.6 ± 0.4	6.92	5.35	6.42
$\Gamma(\psi' \to e^+e^-)$	2.1 ± 0.2	3.42	3.10	2.87
$\Gamma(\psi \to$ hadrons)	57 ± 4	6.49	48.73	58.51
$\Gamma(\psi' \to$ hadrons)		4.54	40.06	37.07
$2^3S_1 \to 1^3P_2$	15 ± 14	39.94	29.87	52.70
$2^3S_1 \to 1^3P_1$	15 ± 8	87.69	79.08	99.62
$2^3S_1 \to 1^3P_0$	16 ± 8	44.89	42.12	49.02
$1^3P_2 \to 1^3S_1$		429.1	451.4	402.1
$1^3P_1 \to 1^3S_1$		193.1	201.1	181.4
$1^3P_0 \to 1^3S_1$		103.6	98.64	97.01

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Resumo

Consideramos neste trabalho a aproximação não-relativística para o sistema $c\bar{c}$ a fim de estudar o espectro da família J/ψ e algumas outras propriedades destas ressonâncias tais como transições hadrônicas, leptônicas e dipolares elétricas. O potencial usado é fenomenológico, com uma parte dependente de spin, dos tipos raiz quadrada e linear. A equação de Schrödinger é resolvida através do método variacional onde se toma por base no-espaço de configuração um conjunto de funções de oscilador harmônico.