Revista Brasileira de Física, Vol. 13, nº 4, 1983

Addition Theorems for Jacobi Functions

J. BELLANDI FILHO and E. CAPELAS DE OLIVEIRA

Instituto de Física, Universidade Estadual de Campinas, Caixa Postal 1170, Campinas, 13100, SP, Brasil

and

H. GERMANO PAVÃO

Departamento de Física e Química, Universidade Federal do Mato Grosso do Sul, Campo Grande, 79100, MS, Brasil

Recebido em 10 de fevereiro de 1984

Abstran We derive addition theorems for Jacobi functions by means of an integral representation for the product of two Jacobi functions.

1. INTRODUCTION

As far as we know, no attempt has been made to derive addition theorems that involves two Jacobi functions of different kinds with different arguments. In this paper we show how to obtain these theorems by using an integral representation for the product of two Jacobi functions of different kinds and different arguments[1]. As a by-product we also get addition theorems for Legendre functions.

2. JACOBI FUNCTIONS

In a recent paper[1] we have shown that for the product of $P_{\nu-n}^{(n-m,n+m)}(x)$ and $Q_{\nu-n}^{(n-m,n+m)}(x')$, the following integral representation is valid

$$2^{-2n}\frac{\Gamma(\nu-n+1)\Gamma(\nu+n+1)}{\Gamma(\nu-m+1)\Gamma(\nu+m+1)}\, P_{\nu-n}^{(n-m,n+m)}(x)\, Q_{\nu-n}^{(n-m,n+m)}(x')$$

$$= (-1)^{\nu-n+1}\,\frac{\Gamma(\nu-n+1)\Gamma(\nu+n+1)}{\Gamma(2\nu+2)}\,\frac{1}{2\pi i}\int_{\infty}^{0^+} dy\,(\text{sh}\, y)^{2\nu+1}\,(\coth y/2)^{2m}$$

$$\{(x+\text{ch}\, y)(x'+\text{ch}\, y)\}^{-\nu-n-1}\;{}_2F_1\left\{\nu+n+1,\ \nu+n+1;\ 2\nu+2;\ \frac{(\text{ch}\, y+1)(\text{ch}\, y-1)}{(\text{ch}\, y+x)(\text{ch}\, y+x')}\right\} \qquad (1)$$

where $1 < x < x' < \infty$. The parameter ν is unrestricted and m and n are restricted by $n+m > 1$ and $n-m > 1$ in order to make the weight function non negative and integrable. The integral in eq.1 means that the path of integration starts at infinity on the real axis, encircles the origin in the positive sense and returns to the starting point.

The hypergeometric function in eq.1 can be written in an integral representation in terms of Bessel function[2]. If we change the cuntour integration into a closed contour around the origin in the positive sense and define

$$\alpha^2 = \frac{(x+1)(x'+1)}{(x-1)(x'-1)} \; ; \; \beta^2 = (x^2-1)(x'^2-1)$$

$$z_1^2 = \alpha\beta \; ; \; z_2^2 = \beta/\alpha \; ; \; z_1 > z_2$$

we can write

$$2^{-2n} \frac{\Gamma(\nu-n+1)\Gamma(\nu+n+1)}{\Gamma(\nu-m+1)\Gamma(\nu+m+1)} P_{\nu-n}^{(n-m,n+m)}(x) \; Q_{\nu-n}^{(n-m,n+m)}(x')$$

$$= \frac{2\pi}{\sin\{n(\nu-n+1/2)\}} \alpha^{-m}\beta^{-n} \oint_{\eta=0} d\eta \; \eta^{-2m+2n-1}$$

$$\times \int_0^\infty dt \; J_{2\nu+1}(2t) \left[\frac{z_1-z_2/\eta^2}{z_1-z_2\,\eta^2}\right]^n Y_{2n} \{(z_1-z_2/\eta^2)^{1/2} (z_1-z_2\eta^2)^{1/2} \; t\} \tag{2}$$

with Re $(\nu-n+1/2) > 0$.

In order to perform the integral over η we chose a particular contour with an unitary ray $\eta = \exp[i\phi/2]$ and we use the Graff addition theorem for the $Y_{2n}(y)$ Bessel function[2]

$$\exp[i\mu\psi] \; Y_\mu(\omega) = \sum_{\ell=-\infty}^{+\infty} Y_{\mu+\ell}(z_1) J_\ell(z_2) \exp[i\ell\phi] \tag{3}$$

where ω, ψ, z_1, z_2, and ϕ are related by means of

$$z_1 - z_2 \cos\phi = \omega\cos\psi$$
$$z_2 \;\text{sen}\;\phi = \omega\;\text{sen}\;\psi$$

The integral over ϕ reproduces a single Kronecker delta, and we get

$$2^{-2n} \frac{\Gamma(\nu-n+1)\Gamma(\nu+n+1)}{\Gamma(\nu-m+1)\Gamma(\nu+m+1)} P_{\nu-n}^{(n-m,n+m)}(x) \quad Q_{\nu-n}^{(n-m,n+m)}(x')$$

$$= \frac{\pi \quad \alpha^{-m} \beta^{-n}}{\sin \pi(\nu-n+1/2)} \int_0^\infty dt \quad J_{2\nu+1}(2t) \; Y_{m+n}(z_1 t) J_{m-n}(z_2 t) \tag{4}$$

with Re $(\nu-n+1/2) > 0$ and $z_1 > z_2$.

To derive the addition theorem for Jacobi functions we multiply both members of eq.4 by $f(x,x')$ $\exp(im\rho)$ where $f(x,x')$ is the weight function

$$f(x,x') = \{(x-1)(x'-1)\}^{(n-m)/2} \quad \{(x+1)(x'+1)\}^{(n+m)/2} \tag{5}$$

and sum over the parameter m to obtain

$$S = \sum_{m=-\infty}^{+\infty} 2^{-2n} \frac{\Gamma(\nu-n+1)\Gamma(\nu+n+1)}{\Gamma(\nu-m+1)\Gamma(\nu+m+1)} \; f(x,x') \; \exp\left[im\rho\right]$$

$$P_{\nu-n}^{(n-m,n+m)}(x) \quad Q_{\nu-n}^{(n-m,n+m)}(x') \tag{6}$$

$$= \frac{\pi}{\sin \pi(\nu-n+1/2)} \int_0^\infty dt \quad J_{2\nu+1}(2t) \sum_{m=-\infty}^{+\infty} Y_{m+n}(z_1 t) J_{m-n}(z_2 t) \; \exp(im\rho)$$

The sum over m in the second member can be performed by the Graff addition theorem. Then we get

$$S = \frac{\pi}{\sin \pi(\nu-n+1/2)} \exp\left[in(\rho+2\psi)\right] \int_0^\infty dt \; J_{2\nu+1}(2t) Y_{2n}(\omega t) \tag{7}$$

with Re $(\nu-n+1/2) > 0$; $z_1 > z_2$ and

$$\exp\left[2i\psi\right] = \frac{z_1 - z_2 \; e^{-i\rho}}{z_1 - z_2 \; e^{i\rho}}$$

$$\omega^2 = (z_1 - z_2 \; e^{-i\rho})(z_1 - z_2 \; e^{i\rho})$$

The integral over t reproduces a Jacobi function[3] of second kind. Then

$$\sum_{m=-\infty}^{\infty} 2^{-2n} \frac{\Gamma(\nu-n+1)\Gamma(\nu+n+1)}{\Gamma(\nu-m+1)\Gamma(\nu+m+1)} \; f(x,x') \; \exp[imp]$$

$$\times \; P_{\nu-n}^{(n-m,n+m)}(x) \; Q_{\nu-n}^{(n-m,n+m)}(x') \tag{8}$$

$$= \exp[in(\rho+2\psi)] \left(\frac{\omega}{2}\right)^{2n} Q_{\nu-n}^{(0,2n)} \{xx' - (x^2-1)^{1/2} (x'^2-1)^{1/2} \cos\rho\}$$

where $1 < x < x' < \infty$; $0 < \psi < \pi/2$ and $\rho \in \mathbb{R}$.

This addition theorem can be easily extended to the domain $|x| < 1$, $|x'| < 1$. If we define $x = \cos\theta$ and $x' = \cos\theta'$, we have

$$\sum_{m=-\infty}^{\infty} (-1)^n \, 2^{-2n} \frac{\Gamma(\nu-n+1)\Gamma(\nu+n+1)}{\Gamma(\nu-m+1)\Gamma(\nu+m+1)} f(\cos\theta, \cos\theta')$$

$$\times \; P_{\nu-n}^{(n-m,\, n+m)}(\cos\theta) \; Q_{\nu-n}^{(n-m,\, n+m)}(\cos\theta') \, . \, \exp[imp] \tag{9}$$

$$= \exp[in(\rho+2\psi)] \left[\frac{1+\cos\gamma}{2}\right]^n Q_{\nu-n}^{(0,2n)}(\cos\gamma)$$

where $\cos\gamma = \cos\theta\cos\theta' + \operatorname{sen}\theta \operatorname{sen}\theta' \cos p$; $0 < \theta + \theta' < \pi$; $0 < \psi < \frac{\pi}{2}$; $\rho \in \mathbb{R}$.

3. LEGENDRE FUNCTIONS

Legendre functions are particular cases of Jacobi functions when $n=0$. Using the well know relation between them, we have for associate Legendre functions in the domain $1 < x < x' < \infty$

$$\sum_{m=-\infty}^{+\infty} (-1)^m \, P_\nu^{-m}(x) \; Q_\nu^m(x') \exp[imp]$$

$$= Q_\nu(xx' - (x^2-1)^{1/2} (x'^2-1)^{1/2} \cos\rho) \tag{10}$$

where $\nu \neq -1,-2 \ldots$ and $\rho \in \mathbb{R}$. For the domain $|x| < 1$ and $|x'| < 1$ with $x = \cos\theta$ and $x' = \cos\theta'$, we have

$$\sum_{m=-\infty}^{\infty} (-1)^m \, Q_\nu^{-m}(\cos\theta) \; Q_\nu^m(\cos\theta') \exp[imp]$$

$$= Q_\nu(\cos\gamma) \tag{11}$$

where $\cos\gamma = \cos\theta\cos\theta' + \text{sen}\,\theta\,\text{sen}\,\theta'\cos\rho$, $0 < \theta' + \theta < \pi$; $\rho \in \mathbb{R}$

REFERENCES

1. J.Bellandi Fº, E. Capelas de Oliveira and H.Germano Pavão, Revista Brasileira de Física, *13*, nº 3, 591 (1983).

2. G.M.Watson, A *treatise on the theory of Bessel functions* (Cambridge University Press, 1966).

3. I.S.Gradshtein and I.M.Ryzhik, *Tables of Integrals, Series and Products* (Academic Press, N.Y., 1980).

Resumo

Deriva-se teoremas de adição para as funções de Jacobi usando-se uma representação integral para o produto de duas funções de Jacobi.