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Weber Transform in a Thermal Conduction Problem

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Abstract We consider a problem of thermal distribution In an infinite medium with a cylindrical cavity of radius a. The Weber transform is used to obtain the solution of the problem. We treat the special case in which the surface **r=a** is crossed by a constant heat flux Q_0 per unit time per unit area.

1. INTRODUCTION

Carslaw and Jaeger[1] have discussed briefly several boundary value problems as examples of the application of the Laplace transformation to cylindrical regions. When the variable is defined in the interval (a,∞) it is convenient to use the Weber transforrn[2]. In previous papers, Battig and Kalla[3], and Battig, Luccioni and Kalla[4] have considered problems of thermal conduction using the Weber trans'form. In this paper we consider another boundary value problem and solve it by means of the Weber transform. The purpose of this note is not only to solve a simple boundary value problem, but to present an exposition of the use of Weber transform which might be useful to tackle other similar problems.

2. THE WEBEIR TRANSFORM

If $f(r)$ is piecewise continuous and absolutely integrable in (a,∞), the Weber transform is defined as

$$W_\nu\left[f(r)\right] = \bar{f}_\nu(\eta) \int_a^\infty r\, f(r)\, Z_\nu(\eta r)\, dr \tag{1}$$

where

$$Z_\nu(\eta r) = J_\nu(\eta r) Y_\nu(\eta a) - Y_\nu(\eta r) J_\nu(\eta a) \tag{2}$$

The functions $J_\nu(x)$ and $Y_\nu(x)$ are the Bessel functions of first and second kind, order ν, respectively.

The inverse of eq. (1) is given by

$$f(r) = \int_0^\infty \bar{f}_\nu(\eta) \frac{Z_\nu(\eta r)}{J_\nu^2(\eta a) + Y_\nu^2(\eta a)} \eta \, d\eta \tag{3}$$

We shall use the following property of this transform[4]

$$\int_a^\infty r\left\{\frac{d^2 f}{dr^2} + \frac{1}{r}\frac{df}{dr} - \frac{\nu^2}{r^2} f\right\} Z_\nu(\eta r) dr = -\frac{2}{\pi} f(a) - \eta^2 \bar{f}_\nu(\eta) \tag{4}$$

3. APPLICATION

We consider the problem of thermal conduction in an infinite medium with a cylindrical cavity of radius a. The interval of variation of radial coordinate is (a,∞). Our problem is to solve the heat equation

$$\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} - k\frac{\partial v}{\partial t} = 0 \tag{5}$$

where v is the temperature, k is the diffusivity of the material, defined as $k = \kappa/\rho c$, where κ, ρ and c are the conductivity density and specific heat of the material. Let'us prescribe the initial and boundary conditions as

$$t = 0 \ , \ r > a, \ v = 0 \tag{6}$$

$$t > 0 \ , \ r = 0, \ k\frac{dv}{dr} = \phi(t) \tag{7}$$

As the temperature function $v(r)$ at $r = a$ is not given we cannot use the Weber transform in a direct form in this problem. We shall transform the equation (5) for the temperature function $v(r)$ to the corresponding equation for the heat flux. As the heat flux f_r is defined as

$$f_r = -k\frac{dv}{dr} \tag{8}$$

eq. (5) becomes

$$\frac{\partial^2 f_r}{\partial r^2} + \frac{1}{r}\frac{\partial f_r}{\partial r} - \frac{1}{r^2} f_r = k\frac{\partial f_r}{\partial t} \tag{9}$$

Multiplying both sides of eq.(9) by $rZ_\nu(\eta r)$ and integrating with respect r from a to ∞ we obtain

$$-\frac{2}{\pi} f_\nu(a) - \eta^2 \bar{f}_\nu(\eta) = k \frac{d\bar{f}_\nu(\eta)}{dt} \tag{10}$$

Using eq. (4), the solution of (10), with the given initial and. boundary conditions (6) and (7), is

$$\bar{f}_\nu(\eta) = -\frac{2}{\pi k} \int_0^t \phi(T)\, e^{\frac{\eta^2}{k}(T-t)}\, dT \tag{11}$$

Applying the inversion formuiae (3) we obtain the general solution of the differentlal equation (9) as

$$f_r(r,t) = -\frac{2}{\pi k} \int_0^t \int_0^\infty \phi(T)\, e^{\frac{\eta^2}{k}(T-t)}\, dT\, \frac{Z_1(\eta r)}{J_1^2(\eta a)+Y_1^2(\eta a)}\, \eta\, d\eta \tag{12}$$

where

$$Z_1(\eta r) = J_1(\eta r) Y_1(\eta a) - Y_1(\eta r) J_1(\eta a)$$

4. A SPECIAL CASE

If we set

$$\phi(t) = Q \quad \text{(a constant)}$$

then the solution (12) reduces to the following form

$$f_r(r,t) = -\frac{2Q}{\pi} \int_0^\infty (1 - e^{-\eta^2 t/k}) \frac{Z_1(\eta r)}{J_1^2(\eta a) + Y_1^2(\eta a)} \cdot \frac{d\eta}{\eta} \tag{13}$$

Defining the new variables

$$\eta a = u \quad , \quad R = \frac{r}{a} \quad , \quad \frac{t}{ka^2} = \tau$$

the solution (13) can be written as

$$f_r(R,t) = -\frac{2Q}{\pi} \int_0^\infty (1 - e^{-u^2\tau}) \frac{Z_1(Ru)}{J_1^2(u) + Y_1^2(u)} \cdot \frac{du}{u}$$

Using the known recurrence formulae

$$\int J_1(Ru)\, dR = -\frac{1}{u} J_0(Ru)$$

and

$$\int Y_1(Ru)dR = -\frac{1}{u} Y_0(Ru)$$

we obtain the temperature

$$v(R,t) = -\int \frac{f_r(R,t)}{k} dR$$

$$= -\frac{2Q}{k\pi} \int_0^{\infty} (1 - e^{-u^2\tau}) \frac{J_0(Ru)Y_1(u) - Y_0(Ru)J_1(u)}{J_1^2(u) + Y_1^2(u)} \cdot \frac{du}{u^2}$$

which reproduces the result given in reference 1, p.338.

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Resumo

Tratamos o problema de distribuição térmica em um meio infinito com uma cavidade cilíndrica de raio a. A transformada de Weber é usada para se obter a solução do problema. Um caso especial é considerado, no qual a superficie $r=a$ é atravessada por um fluxo de calor constante e uniforme.