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n-Soliton Solution to the Einstein Equations

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Abstract The Belinsky and Zakharov n-soliton solution to the Einstein equations for static axially symmetric space-times is generalized.

In a recent paper Belinsky and Zakharov[1] found explicitly a static n-soliton solution to the Einstein equations for axially symmetric space-times by direct integration of the field equations. In a later paper Alekseev and Belinsky[2], using the general method of integration described in Ref.1, succeeded in writting the same solution in a simpler form.

In the present note we generalize the Belinsky Zakharov solution in two different ways: first we take a non-flat "background metric" and secondly we consider different selections of the constants that appear in the one soliton solution.

The vacuum Einstein field equations for the axially symmetric metric

$$ds^2 = f(dr^2 + dz^2) + g_{ab}\, dx^a\, dx^b \tag{1}$$

where f and g_{ab} are functions of r and z only, a, b run from 0 to 1, and $\det g = -r^2$, can be written as

$$(r g_{,r} g^{-1})_{,r} + (r g_{,z} g^{-1})_{,z} = 0 \tag{2}$$

$$\partial_r \ln(rf) = -\frac{1}{4} r \,\mathrm{Tr}(g_{,r} g^{-1}_{,r} - g_{,z} g^{-1}_{,z}) \tag{3}$$

$$\partial_z \ln(rf) = -\frac{1}{2} r \,\mathrm{Tr}(g_{,r} g^{-1}_{,z}) \tag{4}$$

In this letter the matrices g are restricted to the form

$$g = \begin{pmatrix} \exp(\Lambda) & 0 \\ 0 & -r^2\exp(-\Lambda) \end{pmatrix} \tag{5}$$

Note that eq.(5) satisfies the constraint $\det(g) = -r^2$ for any Λ. The one solition solution associated to a "background metric" g_0 of the form (5) admits the four particular cases

$$\exp(\Lambda_1) = \left(\frac{\mu_1}{r}\right)^{\pm 1/2} \left(\frac{\bar{\mu}_1}{r}\right)^{\pm 1/2} \exp(\Lambda_0) \tag{6}$$

with

$$\mu_1 \equiv \alpha_1 - r \pm \left[(\alpha_1 - z)^2 + r^2\right]^{1/2} \tag{7}$$

where α_1 is a complex constant and the bar denotes complex conjugation. By repeating n-times the process indicated in eq.(6) we get the n-soliton solution

$$\exp(\Lambda_n) = \left[\prod_{\ell=1}^{2n} \left(\frac{\mu_\ell}{r}\right)^{\varepsilon_\ell} \exp(\Lambda_0)\right] \tag{8}$$

where the μ_ℓ are functions of the form (7) restricted by $\mu_{s+m} = \bar{\mu}_s . \varepsilon_{s+m} = \varepsilon_s = \pm\frac{1}{2}$ with $1 \leq s \leq n$. It can be proved directly that (8) is a solution to eq.(2), that for matrices like (5) reduces to the usual Laplace equation for an axially symmetric potential Λ_0, i.e., reduces to the integrability condition of the Einstein equations (3) and (4) for the "background metric" g_0. From eqs. (3), (4) and (8) we get

$$\begin{aligned}(\Lambda_n + \ell n f)_{,r} = \frac{r}{2}\Bigg\{ & \sum_{\ell,k} \varepsilon_\ell\, \varepsilon_k \frac{\mu_{\ell,r}\,\mu_{k,r} - \mu_{\ell,z}\,\mu_{k,z}}{\mu_k \mu_\ell} \\ & - \frac{2}{r} \sum_{k,\ell} \varepsilon_k\, \varepsilon_\ell \frac{\mu_{\ell,r}}{\mu_\ell} + 2\sum_\ell \frac{\varepsilon_\ell}{\mu_\ell}(\mu_{\ell,r}\Lambda_{0,r} - \mu_{\ell,z}\,\Lambda_{0,z}) \\ & + \frac{1}{r^2}\Big(\sum_\ell \varepsilon_\ell\Big)^2 - \frac{2}{r}\Big(\sum_\ell \varepsilon_\ell\Big)\Lambda_{0,r} + \Lambda_{0,r}^2 - \Lambda_{0,z}^2 \Bigg\}\end{aligned} \tag{9}$$

and

$$(\Lambda_n + \ell n f)_{,z} = r \left\{ \sum_{\ell,k} \varepsilon_\ell \varepsilon_k \frac{\mu_{\ell,r}}{\mu_\ell} \frac{\mu_{k,z}}{\mu_k} - \frac{1}{r} \sum_{k,\ell} \varepsilon_k \varepsilon_\ell \frac{\mu_{\ell,z}}{\mu_\ell} \right.$$

$$+ \sum_\ell \frac{\varepsilon_\ell}{\mu_\ell} (\mu_{\ell,r}\Lambda_{0,z} + \mu_{\ell,z}\Lambda_{0,r})$$

$$\left. - \frac{1}{r} \left(\sum_\ell \varepsilon_\ell \right) \Lambda_{0,z} + \Lambda_{0,r}\Lambda_{0,z} \right\} \qquad (10)$$

where the indices k and ℓ run from 1 to $2n$. A direct computation shows that the integral of eqs. (9) and *(10)* is

$$\ell n f = \left[\frac{1}{2} \left(\sum_\ell \varepsilon_\ell \right)^2 + \sum_\ell \varepsilon_\ell \right] \ell n r + 2 \sum_{p>q} \sum \nu_p \nu_q \varepsilon_p \varepsilon_q \, \ell n(\mu_p - \mu_q)$$

$$+ 2 \sum_p (\nu_p \varepsilon_p)^2 \, \ell n \mu_p - \sum_\ell \varepsilon_\ell \, \ell n \mu_\ell - \sum_{k,\ell} \varepsilon_k \varepsilon_\ell \, \ell n \mu_\ell$$

$$- \sum_p (\nu_p \varepsilon_p)^2 \, \ell n(\mu_p{}^2 + r^2) - (1 + \sum_\ell \varepsilon_\ell)\Lambda_0$$

$$+ \Omega_0 + \ell n f_0 \qquad (11)$$

where the indices p and q run from 1 to m, m is the numberofdifferent μ_k appearlng in eq.(8), ν_p is the number of times that μ_p appears in eq. (8), and

$$\Omega_0 \equiv \sum_\ell \varepsilon_\ell \int \frac{r}{\mu_\ell} \left[(\mu_{\ell,r}\Lambda_{0,r} - \mu_{\ell,z}\Lambda_{0,z}) dr \right.$$

$$\left. + (\mu_{\ell,r}\Lambda_{0,z} + \mu_{\ell,z}\Lambda_{0,r}) dz \right] \qquad (12)$$

$$\ell n f_0 \equiv \frac{1}{2} \int r \left[(\Lambda_{0,r}^2 - \Lambda_{0,z}^2) dr + 2\Lambda_{0,r}\Lambda_{0,z} \, dz \right] + \ell n \, C_0 \qquad (13)$$

where C, is an Integration constarit. The **existence** of R, and f_0 is guaranteed by the fact that $\ell n(\mu_\ell)$ and A, **satlsfy** Laplace equatlon in cylindrical coordinates.

For the particular Λ_0 given by

$$\Lambda_0 = az\ell nr \tag{14a}$$

$$\Lambda_0 = a\ell nr + bz + c(r^2/2 - z^2) \tag{15a}$$

where a, b, c are constants, the expressions (12) and (13) can be explicitly integrated, yielding

$$\Omega_0 = a \sum_\ell \varepsilon_\ell \{[\bar{\mu}_\ell - 2(\alpha_\ell - z)] \ell nr + \alpha_\ell \ell n\mu_\ell - \mu_\ell + \alpha_\ell - z\} \tag{14b}$$

$$\ell nf_0 = \frac{a^2}{4} [2z^2 r^{-1} - r^2(\ell nr)^2 + r^2 (-\frac{1}{2} + \ell nr)] + \ell n\ C_0 \tag{14c}$$

and

$$\Omega_0 = \sum_\ell \varepsilon_\ell \{a\ell n\mu_\ell + b(\mu_\ell - \alpha_\ell + 2z)$$

$$+ c[\frac{1}{2} r^2 - 2z^2 - (\alpha_\ell + z)\mu_\ell]\} \tag{15b}$$

$$\ell nf_0 = \frac{a^2}{2} \ell nr + az(b-cz) + r^2(ac/2 - b^2/4 + c^2r^2/8$$

$$+ bcz - c^2z^2) + \ell nC_0 \tag{15c}$$

respectively.

The solution presented in reference 1 can be obtalned taking $\bar{\mu}_s = \mu_s$, $\varepsilon_{s+m} = \varepsilon_s = \frac{1}{2}$, $\nu_p = 2$, $m = n$, $\Omega_0 = 0$, and $\ell nf_0 = \ell n\ C_0$ in eq. (11).

The solution presented here is valld for any ε_ℓ, but if $\varepsilon_\ell \neq \pm \frac{1}{2}$ we hsve that the expression (8) does not correspond to a pure soliton. The same is true if $\nu_p > 2$ for some p, because pure soliton solutions are associated to simple poles of the "scatterlng matrix" that allows us to write eqs. (6) and (8). For $\varepsilon_\ell \neq \pm \frac{1}{2}(2k-1)$ we have an essentiel singularity, and for $\nu_p > 2$ we have non-simple poles, in the above nientioned matrix.

REFERENCES

1. V.A. Belinsky and V.E. Sakharov, Sov.Phys.JETP 50, 1 (1979).
2. G.A. Alekseev and V.A. Belinsky, Sov.Phys.JETP *51*, 655 (1980).

Resumo

A solução n-soliton de Belinsky e Zakharov das equações de Einstein para um espaço-tempo com simetria axial é generalizada.