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Integral Representation for the Product cf two Jacobi Functions with Different Order and Arguments

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Abstract New integral representations are obtained for the product of two linear'ly independent Jacobi functions.

1. INTRODUCTION

In a recent paper[1] we have shown how to get an integral representation for the product of two confluent hypergeornetric functions of different orders and argurnents by using an integral representation for the radial Green's function of the isotropic harrnonic oscillator.

An integral representation for the product of two linearly independent Jacobi functions is derived in this paper, and as a by--product we obtain also an integral representation for products of Legendre functions.

2. JACOBI FUNCTIONS

In order to derive an integral representation for the product of two Jacobi functions with different orders and arguments we first write the Jacobi Green's function.

The Jacobi differential operator[2] is written in the form

$$L_x = (1-x^2)\frac{d^2}{dx^2} + \{2m - 2(n+1)x\}\frac{d}{dx} + (\nu-n)(\nu+n+1) \qquad (1)$$

where ν is unrestricted, and with the pararneters m and n restricted by $n+m>1$ and $n-m>1$. These restrictions are needed to rnake the weight function non-negative and integrable.

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The Green's function for this L_X operator, bounded in $1<x<\infty$ is given by

$$G(x,x') = 2^{-2n} \frac{\Gamma(\nu-n+1)\Gamma(\nu+n+1)}{\Gamma(\nu-m+1)\Gamma(\nu+m+1)} P_{\nu-n}^{(n-m;n+m)}(x)\, Q_{\nu-n}^{(n-m;n+m)}(x') \qquad (2)$$

where $x'> X$, and $P_c^{(a,b)}(X)$ and $Q_c^{(a,b)}(X)$ are respectively the first and second solution of the homogeneous Jacobi differential equation.

Introducing in eq. **2** integral representations for the Jacobi functions in terms of Whittaker functions[3]

$$2^{-n} \frac{\Gamma(\nu+n+1)}{\Gamma(\nu-m+1)} P_{\nu-n}^{(n-m;n+m)}(x) = (-1)^{\nu-n+1} \frac{1}{2\pi i} \int_{\infty}^{0+} dt\, e^{-xt}\, t^{n-1}\, W_{m;\nu+\frac{1}{2}}(2t) \qquad (3)$$

$$2^{-n} \frac{\Gamma(\nu+n+1)\Gamma(2\nu+2)}{\Gamma(\nu+m+1)\Gamma(\nu-m+1)} Q_{\nu-n}^{(n-m;n+m)}(x) = \frac{1}{2}\int_0^{\infty} dt\, e^{-xt}\, t^{n-1}\, M_{m;\nu+\frac{1}{2}}(2t) \qquad (4)$$

we obtain for the product of Jacobi functions

$$2^{-2n} \frac{\Gamma(\nu-n+1)\Gamma(\nu+n+1)}{\Gamma(\nu-m+1)\Gamma(\nu+m+1)} P_{\nu-n}^{(n-m;n+m)}(x)\, Q_{\nu-n}^{(n-m;n+m)}(x')$$

$$= \frac{1}{2} \frac{\Gamma(\nu-m+1)\Gamma(\nu-n+1)}{\Gamma(2\nu+2)\Gamma(\nu+n+1)} (-1)^{\nu-n+1} \qquad (5)$$

$$\times \frac{1}{2\pi i} \int_{\infty}^{0+}\int_0^{\infty} e^{-xt-x't'}\, (tt')^{n-1}\, M_{m;\nu+\frac{1}{2}}(2t)\, W_{m;\nu+\frac{1}{2}}(2t')\, dt\,dt'$$

We note that the product of two Whittaker functions in eq. 5 can be identified with the Coulomb radial Green's function[4], when $t'>t$. By a single variable transformation, for $x'> X$, the condition $t'>t$ can be easily reproduced, and the product of Whittaker functions can be written in the form of an integral representation[3]. Thus we get

$$2^{-2n} \frac{\Gamma(\nu-n+1)\Gamma(\nu+n+1)}{\Gamma(\nu-m+1)\Gamma(\nu+m+1)} P_{\nu-n}^{(n-m;n+m)}(x)\, Q_{\nu-n}^{(n-m;n+m)}(x') =$$

$$= \frac{\Gamma(\nu-n+1)}{\Gamma(\nu+n+1)} (-1)^{\nu-n+1} \frac{1}{2\pi i} \int_{\infty}^{0+} \int_0^{\infty} \int_0^{\infty} dt\, dt'\, dy\, (tt')^{n-1/2} (\mathrm{cth}\, y/2)^{2m}$$

$$\times \exp\{-(x+\mathrm{ch}\, y)t - (x'+\mathrm{ch}\, y)t'\} I_{2\nu+1} (2\sqrt{tt'}\, \mathrm{sh}\, y) \tag{6}$$

where $I_\ell(x)$ is the modified Bessel function.

The calculation of the integrals over t and t' are straightforward, and can be performed by means of the hypergeometric function[4]. We obtain

$$2^{-2n} \frac{\Gamma(\nu-n+1)\Gamma(\nu+n+1)}{\Gamma(\nu-m+1)\Gamma(\nu+m+1)} P_{\nu-n}^{(n-m;n+m)}(x)\, Q_{\nu-n}^{(n-m;n+m)}(x') =$$

$$= (-1)^{\nu-n+1} \frac{\Gamma(\nu-n+1)\Gamma(\nu+n+1)}{\Gamma(2\nu+2)} \frac{1}{2\pi i} \int_{\infty}^{0+} dy\, (\mathrm{sh}\, y)^{2\nu+1} (\mathrm{cth}\, y/2)^{2m}$$

$$\times \{(x+\mathrm{ch}\, y)(x'+\mathrm{ch}\, y)\}^{-\nu-n-1} {}_2F_1 \left\{\nu+n+1; \nu+n+1; 2\nu+2; \frac{(\mathrm{ch}\, y+1)(\mathrm{ch}\, y-1)}{(\mathrm{ch}\, y+x)(\mathrm{ch}\, y-x)}\right\} \tag{7}$$

where $1 < x < x' < w$.

Ttie right hand side of this expression is symmetric with respect to x and x', and in order to symmetrize also the first term we introduce e symmetric expression for the product of Jacobi functions in the following way

$$\left\{P_{\nu-n}^{(n-m;n+m)}(x) \quad Q_{\nu-n}^{(n-m;n+m)}(x')\right\} =$$

$$= \frac{1}{2} \left[P_{\nu-n}^{(n-m;n+m)}(x)\, Q_{\nu-n}^{(n-m;n+m)}(x') + P_{\nu-n}^{(n-m;n+m)}(x')\, Q_{\nu-n}^{(n-m;n+m)}(x)\right] \tag{8}$$

By a single variable transformation on the cornplex plane eq. 7 can be easily transformed into

$$\frac{\Gamma(\nu-n+1)\Gamma(\nu+n+1)}{\Gamma(\nu-m+1)\Gamma(\nu+m+1)}\left\{P_{\nu-n}^{(n-m;n+m)}(x)\ Q_{\nu-n}^{(n-m,n+m)}(x')\right\} =$$

$$= \alpha^{-\frac{1}{2}(n+m)}\ \frac{1}{2\pi}\oint_{\sigma=0} d\sigma\ e^{-i(n+m)\sigma}\ (1-\sqrt{\alpha}\, e^{i\sigma})^{2n}\ Q_{\nu-n}^{(0;2n)}\ (\mathrm{ch}\ \gamma) \qquad (9)$$

where $\alpha = (x+1)(x'+1)/(x-1)(x'-1)$ and $\mathrm{ch}\ \gamma = xx' - (x^2-1)^{1/2}(x'^2-1)^{1/2}$ $\cos\sigma$, and the integral contour is a closed contour around the origin, $\sigma=0$.

In a similar way we can extend this integral representationto thedomain $|x| < 1$, $|x'| < 1$. Defining $x = \cos\theta$ and $x' = \cos\theta'$, we have

$$\frac{\Gamma(\nu-n+1)\Gamma(\nu+n+1)}{\Gamma(\nu-m+1)\Gamma(\nu+m+1)}\left\{P_{\nu-n}^{(n-m;n+m)}(x)\ Q_{\nu-n}^{(n-m;n+m)}(x')\right\} =$$

$$= (\tan\theta/2\ \tan\theta'/2)^{-n-m}$$

$$\times\frac{1}{2\pi}\oint_{\sigma=0} d\sigma\ e^{-i(n+m)\sigma}\ (I-\tan\theta/2\ \tan\theta'/2\ e^{i\sigma})^{2n}$$

$$\times\ Q_{\nu}^{(0;2n)}(\cos 0\ \cos\theta' + \mathrm{sen}\,\theta\ \mathrm{sen}\ 6'\ \cos o) \qquad (10)$$

with $0 < \theta < \theta' < \pi$.

3. LEGENDRE FUNCTIONS

Legendre functlons are a particular case of Jacobi functions. For n=O we have the associated Legendre function. The relation between them are well known[4], and for the domain $1 < x < x' < \infty$, we have

$$\{P_{\nu}^{m}(x)\ Q_{\nu}^{-m}(x')\} = \frac{1}{2\pi}\oint_{\sigma=0} d\sigma\ \ e^{-im\sigma}\ \ Q_{\nu}(\mathrm{ch}\,\gamma) \qquad (11)$$

with $\mathrm{ch}\ \gamma = xx' - (x^2-1)^{1/2}(x'^2-1)^{1/2}\ \cos o$.

For the domain $|x| < 1$ and $|x'| < 1$, the integral representation for associated Legendre polynomials is

$$\{P_\nu^m(\cos\theta)\; Q_\nu^{-m}(\cos\theta')\} = \frac{1}{2\pi} \oint_{\sigma=0} d\sigma \; e^{-im\sigma}\, Q_\nu(\cos\gamma) \qquad (12)$$

with $\cos\gamma = \cos\theta\,\cos\theta' + \text{sen}\,\theta\,\text{sen}\,\theta'\cos\sigma$ and $0 < \theta < \theta' < \pi$.

When the parameter is put $m=0$ in these expressions, we obtain integral representations for products of Legendre functions.

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Resumo

Obtemos novas representações Integrais para o produto de duas funções de Jacobi linearmente independentes.