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An Integrability Condition for Gauge Field Copies

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Abstract We show that $[\Omega \wedge \theta] = 0$, where θ defines an integrable subbundle in the tangent bundle T.P. to the principal bundle P associated to a gauge field theory on a general n-dimensional manifold, is a necessary and sufficient condition for the existence of potential ambiguities in that gauge theory.

1. INTRODUCTION

For mathematicians a gauge field theory is described on a principal fiber bundle $P(M,G)$ over a finite-dimensional smooth n-manifold M and with a finite-dimensional semi-simple Lie group G as its fiber. The gauge potential is identified with a connection form ω on the bundle, that is, an $L(G)$-valued smooth I-form over $P(M,G)$, where $L(G)$ is the group's Lie algebra. The gauge field is then ω's exterior covariant derivative, that is,

$$\Omega = d\omega + \frac{1}{2} [\omega \wedge \omega] \tag{1.1}$$

according to one of Cartan's structure equations[1].

Suppose now that Ω can be derived from two different potentials ω and ω' on $P(M,G)$. Are ω and ω' gauge-related potentials? Is there any physically meaningful map that sends ω over ω'? Given an arbitrary gauge field Ω on $P(M,G)$, how can we check whether that field has a potential ambiguity? The gauge field copy problem answers those questions[2] and characterizes in a complete way all potentials for a given gauge field.

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However it has been long known that there is a simple necessary criterion that a gauge field R must satisfy in order to have any potential ambiguity: the linear functional system

$$[\Omega \wedge \theta] = 0 \tag{1.2}$$

must have nontrivial solutions.

Here 0 is an $L(G)$-valued tensorial 1-form on the bundle. In a local coordlnate system eq. (1.2) becomes

$$\varepsilon^{\mu\nu\rho\ldots\sigma}[\Omega_{\mu\nu},\theta_\rho] = 0 \tag{1.3}$$

which in four dimensions can be written as

$$[\varepsilon^{\mu\nu\rho\sigma}\,\Omega_{\mu\nu},\theta_\rho] = [{}^*\Omega^{\rho\sigma},\ \theta_\rho] = 0 \tag{1.4}$$

It has been recently noticed[3] that eq. (1.4) is not a sufficient condition for the existence of a potential ambiguity associated to the field R, for $G = SU(2)$ or $SO(3)$. We show here that the correct necessary and sufficient conditions for a gauge field R to have any potential ambiguity are slightly stricter than eq. (1.2): the field must satisfy $[\Omega \wedge \theta] = 0$ aad the geometric distribution thus defined must be an integrable distribution' is the sense of Frobenius' theorem[5]. Furthermore, if Ω satisf ies also another condition,

$$[\Omega,\rho] = 0 \tag{1.5}$$

where ρ is a (possibly local) $L(G)$-valued smooth function, then (and only then) Ω will have ambiguous potentials that are (locally at least) gauge- equivalent. In the general case however R will not have gauge--equivalent ambiguous potentials; they will be A-equivalent, in the sense of Einstein's A-transformation[6].

The present situation parallels similar conditions on geometric structures applied to Physics: for example, let N be a smooth finite--dimensional manifold and let a be a 2-form on N. If X is á smooth vectorfield on N then X is a cross-section of the tangent bundle T.N. We then define

$$R_\alpha = \{X,\ i_X\,\alpha = 0\ \} \tag{1.6}$$

to be a's *characteristic bundle.* Then $da = 0$, and a is a (degenerate) symplectic form if and only if R_a is integrable, that is, if R is the tangent bundle $T.Q$ to a submanifold $Q \subset N$; we then have the bundle inclusion $R_\alpha = T.Q \subset T.N$. As in the gauge field copy problem we have here a linear degeneracy condition,

$$i_X \alpha = 0 \tag{1.7}$$

whose nullspace must be integrable in a physically meaningful situation, for that condition lies at the base of time-dependent Hamilton--Jacobi theory.

As another example we may quote the distínction between almost-complex and complex manifolds[8]. A complex manifold is an almost complex manifold with an integrable tangent bundle associated to the almost complex operator J that maps "real" vectorfields on "imaginary" vectorfields. Again we can define the relevant distributions with the help of a linear degeneracy condition such as in eqs. (1.2) or (1.7); an almost complex manifold can be endowed with an analytical structure compatible with the almost complex structure provided that the almost complex structure be integrable in the sence described above.

We believe that the interplay between 'quasi-copied fields' - which are those fields that satisfy $[\Omega \wedge \theta] = 0$ - and 'copied fields' - those for which θ is integrable - can be of physical import, as copied fields will be gauge fields with a vanishing torsion tensor

$$\tau = d\theta + \frac{1}{2}[\theta \wedge \theta] + [\omega \wedge \theta] \tag{1.8}$$

very much like the situation in the almost complex/complex case, where integrability also depends on a vanishing torsion tensor.

The parallel is even closer than it may seem at first sight: we have shown that Kaluza-Klein theories with an analytical Kaehler structure compatible with the Kaluza structure necessarily imply that all gauge fields involved be copied gauge fields[9].

The following conventions are used here: all objects are supposed to be smooth unless otherwise specified. Vectorfields are denoted by $X, Y, Z, \ldots$ Scalar constants by $k, m, \ldots$ Differential forms by Greek lowercase letters; the single exception is thecurvature form (the gauge field), always denoted by R.

Lie algebra commutators are $[\ ,\ ]$. The algebra of $L(G)$-valued differential forms has a natural graded Lie algebra structure that arises out of the combination of $\wedge$ and $[\ ,\ ]$; its (graded) commutator is written $[\alpha \wedge \beta]$. We notice that for an $L(G)$-valued p-form α and an $L(G)$-valued q-form β, we have:

$$[\alpha \wedge \beta] = (-1)^{1+p+q} [\beta \wedge \alpha] \tag{1.9}$$

2. A PRIMER ON INTEGRABILITY CONDITIONS

Suppose that we are given a family of linearly independent vectorfields $X_{(i)}$, $i = 1,2,\ldots, p \leq n$, on a n-dimensional differentiable manifold M. Such a family and its (constant coefficient) linear combinations generate a p-dimensional vectorspace which we denote by $(X_{(i)})$. Obviously $(X_{(i)}) \subset T.M$, that is, $(X_{(i)})$ is a subvectorspace in the tangent space to M, or, in a more technical language, $(X_{(i)})$ is a subbundle of the tangent bundle T.M (for a review of these and related geometric concepts see ref.10. $(X_{(i)})$ is also usually called a geometric *distribution* in the literature — not to be confused with the more familiar Schwartz distributions such as the Dirac delta[11].)

We can now ask the following question: is $(X_{(i)})$ the tangent space (tangent bundle) to a submanifold $N \subset M$? The answer is no, in general. The question is settled by the *Frobenius Integrability Theorem*, stated below:

Theorem 2.1. (Frobenius, 1st version) $(X_{(m)}) \subset T.M$ is tangent to a submanifold $N \subset M$ iff* $[X_{(i)}, X_{(j)}] \in (X_{(m)})$, for 1 i, j. If that conditiori is satisfied, $(X_{(m)})$ is called an *integrable distribution.*

For the proof see ref.12. $[X_{(i)}, X_{(j)}]$ denotes the Lie bracket of two vectorfields. Vectorfields on manifolds are essentially directional derivatives, or infinitesimal displacement operators; they can be thus represented in local coordinate systems:

$$X = X^{\mu}\,\partial_{\mu} \tag{2.1a}$$

$$Y = Y^{\nu}\,\partial_{\nu} \tag{2.1b}$$

And their Lie bracket,

$$[X,Y]^{\nu} = X^{\mu}\,\partial_{\mu} Y^{\nu} - Y^{\mu}\,\partial_{\mu} X^{\nu} \tag{2.2}$$

---

* "iff" is common abbreviation for "if and only if".

Eq.(2.2) looks a bit like a Poisson bracket. In fact Lie and Poisson brackets are closely related; a Poisson bracket is the Lie bracket of two gradients on phase space; for more details see references 13 and 14.

Geometric distributions are thus related to a class of systems of partial dlfferential equations; since we leave boundary conditions unspecified, an integrable geometric distribution will determineafamily of p-dirnensional subrnanifolds of M.Such a farnily is called a p-*dimensional foliation* on M. Each submanifold in the foliation will be an *immersed* submanifold of M; it will not in general be an *embedded* submanifold. That is to say, it will fit nicely into M only locally; when we patch up everything together the global situation won't be as well behaved as the local one.

We can aiso define geometric distributions on M with the help of differential forms: let $\omega^{(k)}$, $k = 1,2,\dots,n-p$, be a farnily of linearly independent 1-forms on M. We define a distribution (X) as follows:

$$(X) = \{X \in T.M,\ \omega^{(k)}(X) = 0\} \tag{2.3}$$

That is to say, (X) is the *nullspace* of the forms $\omega^{(k)}$. We also say that the $\omega^{(k)}$ *kill* (send over zero) all vectors in the distribution (X).

Integrability via differential forms is given by:

Theorem 2.2. (Frobenius, 2nd version) $(X) \subset T.M$ defined by (2.2) is an integrable distribution iff there exists a collection of $(n-p)^2$ I-forrns $\alpha^{(k)}_{(m)}$ on M such that

$$d\omega^{(k)} = -\alpha^{(k)}_{(m)} \wedge \omega^{(m)} \tag{2.4}$$

Eq.(2.3) is very famil iar to theoretical physicists: we can write it up **in a local coordinate system as:**

$$\partial_\mu \omega^{(k)}_\nu - \partial_\nu \omega^{(k)}_\mu + \alpha^{(k)}_{\mu(m)} \omega^{(m)}_\nu - \alpha^{(k)}_{\nu(m)} \omega^{(m)}_\mu = 0 \tag{2.5}$$

That is, it asserts that the exterior covariant derivative of the vector-valued I-form $\omega = (\omega^{(k)}_\mu)$ vanishes in order that the associated distribution be integrable. A still more famil iar interpretation for the $\alpha^{(k)}_{(m)}$ readily follows, as Theorem 2.2 is equivalent to:

Thereom 2.3. (Frobenius, 3rd version) (X) given by eq. (2.3) is integrable provided that there exists a $(n-p) \times (n-p)$ functionalmatrix

$u = (u^{(m)}_{(n)})$ such that

$$\omega^{(k)} = u^{(k)}_{(m)} \, d\beta^{(m)} \tag{2.6}$$

We now apply eq. (2.6) to eq. (2.4) and get, after a few simple calculations

$$(u^{-1} \alpha u + u^{-1} du) \wedge d\beta = 0 \tag{2.7}$$

where

$$u = (u^{(k)}_{(m)}) , \quad \alpha = (\alpha^{(k)}_{(m)}) , \quad d\beta = (d\beta^{(k)})$$

As the expression inside brackets in eq. (2.7) is a $(n-p)\times(n-p)$ matrix that acts on a vectorvalued form, and as degeneracies are excluded, a simple linear algebra reasoning leads to the consequence that

$$u^{-1} \alpha u + u^{-1} du = 0 \tag{2.8}$$

α is a "vacuum potential", a "pure gauge".

The analysis that follows Theorem 2.3 is crucial to our exposition in the coming section. In fact, eq. (2.8) asserts that integrable distributions have vanishing curvatures and are thus flat in the usual sense.

A few remarks will provide further clarification on these ideas:

(i) Due to eq. (2.6) an integrable distribution defined by a set of forms $\omega^{(k)}$ can be also represented by another set $d\beta^{(k)}$. As (X) is characterized by $d\beta^{(k)}(X) = 0$, we can integrate this set of equations and get a system

$$\beta^{(k)} = \text{const.} \tag{2.9}$$

In general such an integration is only local. (2.9) defines a local holonomic coordinate system, and the $\beta^{(k)}$ are coordinate functions in that coordinate system.

(ii) The vector field counterpart to eq. (2.9) is simply

$$[Y_{(i)}, Y_{(j)}] = 0 \tag{2.10}$$

A holonomic coordinate vectorfield is then $Y_{(i)} = \partial/\partial x^{(i)}$.

3. THE FIELD COPY PROBLEM AS AN INTEGRABILITY PROBLEM

We show here that the field copy problem reduces to an integrability condition. In the next Section we show that this integrability condition is equivalent to a linear degeneracy condition as the one given by eq. (1.7). The main line of reasoning goes as follows: we recognize the field copy problem as an integrability problem and then apply an adequate version of Frobenius' Theorem to it. We are then left with a possible classification for field copies: (i! locally at least gauge -equivalent copied potentials? (ii) infinitesimally copied potentials and (iii) discrete copies. The main proofs are quoted in full.

We suppose that we are given here a smooth principal bundle $P(M,G)$ over a smooth n-dimensional real manifold M. G is a p-dimensional semi-simple Lie group. If we endow P with a connection described by the connection form ω then the associated curvature is given by Cartan's structure equation

$$\Omega = d\omega + \frac{1}{2}[\omega \wedge \omega] \tag{3.1}$$

If ω and ω' are different connection forms for the same curvature form Ω we must have

$$\Omega = d\omega + \frac{1}{2}[\omega \wedge \omega] = $$

$$= d(\omega+\theta) + \frac{1}{2}[(\omega+\theta) \wedge (\omega+\theta)] , \tag{3.2}$$

where we have written $\omega' = \omega + \theta$, wherefrom one gets that

$$\tau = d\theta + \frac{1}{2}[\theta \wedge \theta] + [\omega \wedge \theta] = 0 \tag{3.3}$$

If we now introduce a (possibly local, over an open $\pi^{-1}U$, $U \subset M$ being a nonvoid open set) auxiliary connection form[15] $\omega^0 = w + \frac{1}{2}\theta$. then eq. (3.3) becomes

$$\tau = d(\omega^0)\theta \equiv_{Def} d\theta + [\omega^0 \wedge \theta] \tag{3.4}$$

Eq. (3.4) is clearly an integrability condition such as the one in Theorem 2.2; if θ is seen as a generalized solder form[16], eq. (3.4) defines a torsion tensor w.r.t. the connection ω^0. The field copy problem amounts to the solution of eq. (3.4) and to the classification of all connection forms ω^0 that (nontrivially) satisfy it. If we choose a local Lie algebra basis on the bundle, eq. (3.4) becomes:

$$d\theta^a + c^a{}_{bc}\, \omega^{0b} \wedge \theta^c \tag{3.5}$$

where $c^a{}_{bc}$ are the structure constants. We have here $k \leq p$ 1-forms on a $(n+p)$ manifold; we can then apply Theorem 2.2 to eq. (3.5). We must only notice that eq. (3.4) is an equivariant equation, that is, it does not depend on the Lie group degrees of freedom when we factor out gauge transformations. Eq. (3.5) splits into

$$\begin{cases} d\theta^a + c^a{}_{bc}\, \omega^{0b} \wedge \theta^c\,, & a \leq k \quad (3.6a) \\ c^a{}_{bc}\, \omega^{0b} \wedge \theta^c = 0\,, & k < a \quad (3.6b) \end{cases}$$

Now Theorem 2.3 says that there is a local linear transformation that sends each θ^a over an exact form $d\beta^a$; since everything is equivariant, that transformation extends to a local gauge transformation that maps eqs. (3.6) over

$$[\omega^0 \wedge \theta] = 0\,, \tag{3.7a}$$

$$\theta = d\beta \tag{3.7b}$$

(Here we have denoted the transformed objects by the same symbols; the two systems in eqs. (3.6) fuse into eq. (3.7a)). As now we have in the new gauge that $\omega = \omega^0 - \frac{1}{2}\, d\beta$, (3.7a) becomes

$$[\omega \wedge d\beta] = \frac{1}{2}\, [d\beta \wedge d\beta] \tag{3.8}$$

We can split eq. (3.8) into two situations: either eq. (3.8) holds in full,

$$[\omega \wedge d\beta] = \frac{1}{2}\, [d\beta \wedge d\beta] \neq 0 \tag{3.9a}$$

or we have a degeneracy,

$$[\omega \wedge d\beta] = 0 \tag{3.9b}$$

$$[d\beta \wedge d\beta] = 0 \tag{3.9c}$$

We notice that if k is a constant, $k\, d\beta$ also satisfies eqs. (3.9b) and (3.9c), granted that $d\beta$ satisfies them. Conversely, if we define an infinitesimal copy by setting $\theta' = k\theta$, $k > 0$, $k^2 \cong 0$, eq.(3.3) becomes approximately

$$d\theta' + [\omega \wedge \theta'] = 0 \qquad (3.10)$$

which immediately leads to

$$[\omega \wedge d\beta'] = 0 \qquad (3.11a)$$

$$[d\beta' \wedge d\beta'] = 0 \qquad (3.11b)$$

We thus call copied connections that satisfy (3.9b-3.9c) (or eqs.(3.11)) *infinitesimally* **copied connections.** If however those conditions are not fulfilled we can always split the connection form ω in eq. (3.9a) into $\omega = \bar{\omega} + \frac{1}{2} d\beta$, wherefrom (3.9a) entails that

$$[\bar{\omega} \wedge \beta] = 0 \qquad (3.12)$$

$\bar{\omega}$ is immediately seen to be an infinitesimally copied connection; the original connection is $\omega = \bar{\omega} + \frac{1}{2} d\beta$ and its unique copy is $\omega' = \bar{\omega} - \frac{1}{2} d\beta$. In either case it is immediately seen that the gauge potential and its copy or copies are related by an Abelian gauge transformation that slightly generalizes Einstein's λ-transformation[17].

There remains one final question: when does a curvature form posses locally at least two different but gauge equivalent potentials? A necessary condition is simple: if w and ω' are potentials for the same curvature R, and if, say, $\omega' = u^{-1}\omega u + u^{-1}du$ then $u^{-1}\Omega u = R$. Thus there will be gauge transformation generators, that is to say, Lie algebra valued functions such that $[\Omega,\rho] = 0$. Is that condition also sufficient? Suppose that there exists a non-trivial Lie algebra valued function ρ such that one has $[\Omega,\rho] = 0$. Then R has at least one potential ω such that $[\omega,\rho] = 0$. For an infinitesimal constant k ($k>0$, $k^2 \cong 0$) and from eq.(3.10) one gets that $8 = d\rho$. All such solutions span a centralizer for the Ambrose-Singer holonomy group, whose generators are all values of the $\Omega(X,Y)$. We thus get that there must be a nontrivial ρ such that $[\Omega,\rho] = 0$.

We can summarize as follows the preceding results:

Proposition. Given a connection form ω it will admit a copy if and only if it admits at least one vanishing torsion tensor $\tau=0$ given by eq. (3.3).

It will admit a single copy provided that it satisfies eq.

(3.9a). In that case both copies are not gauge equivalent

It will admit an infinite family of copies if it satisfies eqs.(3.9b) and (3.9c).

It will finally admit gauge-equivalent copies provided its curvature nontrivially satisfies $[\Omega,\rho] = 0$.

4. GAUGE FIIELD COPIES IFF $[\Omega \wedge \theta] = 0$, θ INTEGRABLE

There is a disadvantage in the preceding criteria for the existence of group field copies: one would like to recognize the possibility of copies in the fields themselves, and not in their potentials. Potentials are notoriously clumsy to deal with due to their nonhomogenous transformation law under gauge actions, while fields behave in a much neater way under the same group. The condition for existence of gauge copies is given by eqs. (3.7); do we have a similar linear' degeneracy condition involving gauge fields?

We wish to prove here the following theorem:

Theorem. Ω has a potential ambiguity if and only if $[\Omega \wedge \theta] = 0$, θ a nontrivial, integrable, equivariant 1-form.

The proof is divided into several steps. We first show the necessity:

Lemma 4.1. If R has a potential ambiguity then $[\Omega \wedge \theta] = 0$ defines a nontrivial integrable distribution.

Proof: Ω must satisfy at least two different differential Bianchi conditions:

$$d\Omega + [\omega \wedge \Omega] = 0 \qquad (4.1a)$$

$$d\Omega + [(\omega + d\beta) \wedge \Omega] = 0 \qquad (4.1b)$$

If we subtract the first equationabove from the second, we get the desired result. We have used here the fact that $\omega' = \omega + d\beta$ in a particular gauge.

Sufficiency is much harder. We start with:

Lemma 4.2. $[\Omega \wedge \theta] = 0$, θ integrable, is equivalent to

$$d[\omega \wedge d\beta] + [\omega \wedge [\omega \wedge d\beta]] = 0 \qquad (4.2)$$

at the gauge where $\theta = d\beta$.

Proof: $[\Omega \wedge d\beta] = 0$ iff (= if and only if)

$$[(d\omega + \frac{1}{2}[\omega \wedge \omega]) \wedge d\beta] = 0$$

(due to Cartan's structure equation). We then get

$$[d\omega \wedge d\beta] + \frac{1}{2}[[\omega \wedge \omega] \wedge d\beta] = 0$$

And then

$$d[\omega \wedge d\beta] + \frac{1}{2}[[\omega \wedge \omega] \wedge d\beta] = 0$$

Finally, due to the Jacobi identity, we get our result:,

$$d[\omega \wedge d\beta] + \frac{1}{2}[\omega \wedge [\omega \wedge d\beta]] = 0$$

We now must look for solutions of

$$d\sigma + [\omega \wedge \sigma] = 0 \tag{4.3}$$

where σ is a k-form, and $d(\omega)$ a and equivariant form. We settle that question with the following result:

Lemma 4.3. There is a local gauge transformation u: $U \times G \rightarrow U \times G$ such that $d(\omega)\ \sigma = 0$ is mapped over

$$[\omega \wedge \sigma] = 0 \tag{4.4a}$$

where a is integrable

$$\sigma = d\xi \tag{4.4b}$$

Proof: σ Is an equivariant $L(G)$-valued k-form. If we use a local $L(G)$ frame, it can be expanded as $\sigma = U^a{}_b\, d\xi^b$, where the $d\xi^b$ are k-1 $L(G)$--valued forms.

It is immediately checked that σ will be equivariant provided that $\xi = \xi^b X_b$ be equivariant (where the X_b are Lie algebra generators.) The above conditions can be written in matrix notation as:

$$\sigma = U\, d\xi = Ad(u)\, d\xi = u^{-1}\, d\xi\, u \tag{4.5}$$

whereu is a (local) gauge transformation. Eqs. (4.5) result from the following two facts: first, as σ is equivariant, that property will be preserved only under the linear action of the gauge group. Second, as σ is $L(G)$-valued, such an action must be an adjoint (Ad) action.

If we now substitute eq. (4.5) into $d(\omega)\sigma = 0$, we get eqs. (4.4).

We have already shown that $[\Omega \wedge d\beta] = 0$ is equivalent to $d[\omega \wedge d\beta] + [\omega \wedge [\omega \wedge d\beta]] = 0$. Lemma 4.3 allows us to solve that equation: we impose the condition that $[\omega \wedge d\beta]$ be an exact form, and then substitute it into eq. (4.4a).

We first notice that all exact forms that solve eq. (4.3) will appear exact at the same gauge (or at the same set of gauges), which will be "holonomic" gauges for the connection w. In those gauges some components of ω will be made to vanish. Thus we can write that, for a l-form σ,

$$[\omega \wedge d\beta] = [d\sigma \wedge d\beta] \qquad (4.6)$$

We can split $\omega = \tilde{\omega} + d\sigma$, and because of eq. (4.6), we must have:

$$[\tilde{\omega} \wedge d\beta] = 0 \qquad (4.7)$$

where $\tilde{\omega}$ is supposed to be no cocycle. That decomposition, taken into eq. (4.4a) leads to

$$[\tilde{\omega} \wedge [d\sigma \wedge d\beta]] + [d\sigma \wedge [d\sigma \wedge d\beta]] = 0 \qquad (4.8)$$

Since the First term is no cocycle and as the 2nd term is an exact form we must have

$$\begin{cases} [\tilde{\omega} \wedge [d\sigma \wedge d\beta]] = 0 & (4.9a) \\ [d\sigma \wedge [d\sigma \wedge d\beta]] = 0 & (4.9b) \end{cases}$$

We must now classify solutions for the above equations, granted that $\tilde{\omega}$ must satisfy eq. (4.7), that is,

$$[\tilde{\omega} \wedge d\beta] = 0$$

We notice that eqs. (4.9) form a partial first order differential system of equetions for a; it is an equivariant, overdetermined system. As da is already present here as an exact form, it must satisfy the flatness condition given by eq. (2.7) w.r.t. $\tilde{\omega}$, that is, one must have

$$[\tilde{\omega} \wedge d\sigma] = 0 \qquad (4.10)$$

Now, since $\omega = \tilde{\omega} + d\sigma$, ω is immediately recognized as a copied connection. In particular, as eq,(4.9b) is equivalent to

$$[[d\sigma \wedge d\sigma] \wedge d\beta] = 0 \tag{4.11}$$

via the Jacobi identity, we may check that $[d\sigma \wedge d\sigma] = 0$ gives continuous copies, and that $d\sigma = k\, d\beta$, k a constant, gives discrete copies.

We have thus proved that w is a copied potential.

5. CONCLUSIONS

We have shown that integrability of the bundle θ described by the linear homogeneous system

$$[\Omega \wedge \theta] = 0 \tag{5.1}$$

is equivalent to the vanishing of the torsion-like tensor

$$\tau = d\theta + \frac{1}{2}[\theta \wedge \theta] + [\omega \wedge \theta] \tag{5.2}$$

and it thus leads to the existence of potential ambiguities in gauge fields. Possible meanings for the tensor τ given by eq.(5.1) have been recently discussed[18]. A very striking example relates that object to the extension of analytical structures to general phase spaces in Classical Mechanics; one is left with a theory that has some of the flavor of a Kaluza-Klein-type unified theory[19].

Another remarkable development lies in the situation of copied fields in the space of all G-gauge fields. Singer has shown[20] that reducible unitary fields belong to a stratified manifold structure inside gauge field space (a stratified manifold is a union of submanifolds of a given manifold such that a n-k-p dimensional submanifold is in the boundary of a n-k dimensional manifold in the collection). We can show that the Singer structure is also the way gauge fields with gauge-equivalent copies appear in field space, and eq.(5.1) allows us to describe a stratification for all fields that globally satisfy a condition like it[21]. But what about fields that only locally do satisfy eq.(5.1)? These fields belong to a much more complicated domain in gauge field space; it is a fractal domain in the sense of Mandelbrot[22], a kind of Devil's ladder[23] where every stratification is bounded by still another stratification, and so on ad infinitum, but everything kept inside a nowhere dense, a "zero-volume" region of gauge field space. That structure is a dynamical system attractor for gauge field theories with broken symmetries; it is also a very general phenomenon that may manifest itself from Physics to Mathematical Ecology and sociobiological models[24]. Work on it is now in progress[25].

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Resumo

Mostramos que $[\Omega \wedge \theta] = 0$, onde θ define um subfibrado integrável do fibrado tangente T.P ao fibrado principal P que se associa a uma teòria de "gauge" sobre uma variedade n-dimensional genérica, é uma condição necessária e suficiente para a existência de ambiguidades de potential naquela teoria de gauge.