Revista Brasileira de Física, Vol. 13, nº 3, 1983

The Spherical Model in a Random External Field

J.F. PEREZ and **W.F. WRESZINSKI**

Instituto de Física, Universidade de São Paulo, Caixa Postal 20516, São Paulo, 01000, SP, Brasil

Recebido em :30 de novembro de 1982

Abstract We calculate the free energy of the spherical model in a random external field and prove the absence of ferromagnetism if the dimension is $\nu \leq 4$, as well as existence of a ferromagnetic phase transition if $\nu \geq 5$ whenever the variance of the external field is sufficiently small.

1. INTRODUCTION

Schuster[1] gave an appealing argument for the absence of a ferromagnetic phase transition in the classical *xy* model with random external field for $\nu \leq 4$. His argument consists of a combination of the Mermin Wagner proof of absence of a phase transition with the replica trick[2]. The applicability of the latter is, however, doubtful for reasons which were analysed in detail in reference 2.

In reference 3 we calculated the free energy of the mean spherical model in a random external magnetic field by a method which also enabled a parallel proof that the convergence of the various quantities involved is with probability one - a result which is essential to account for the reproducibility of the measurements performed on random systems. We also refer to reference 4 for similar results obtained in a more abstract framework.

In this note we compute the free energy of the mean spherical model in a standard and straightforward fashion which does not, however, allow a direct proof of "reproducibility". We therefore called it "very mean", but felt that nevertheless the simplicity of the derivation might be of some pedagogic value, specially for newcomers in this field.

We should like to add that the spherical model is a prototype for phase transitions of *continuous systems* because it is the "limit" $N\to\infty$ of the continuous N-vector models[5], and it saturates the infrared bounds for continuous models of references 6 and 7. It is therefore, as far as we know, the only exactly soluble model providing strong support for the assertion[8] that the lower critical dimension of continuous systems in a random magnetic field is four.

Finally, we rernark that the phase transition in the free Bose gas and in the spherical model is very similar[9]. The free Bose gas in a randorn external field was studied in ref. 10, an important precursor.

2. RESULTS

We define the random external field in terms of a collection of independent (not necessarily Gaussian as in ref. 1) random variables $h(x)$ of mean zero and variance a (independent of x) in some probability space. If ρ denotes the probability measure, and $\langle \cdot \rangle_\rho$ expectations with respect to p, these conditions may be written

$$\langle h(x) \rangle_\rho = 0 \qquad \forall\, x \in \mathbb{Z}^\nu \tag{1}$$

and

$$\langle h(x) h(y) \rangle_\rho = \sigma^2 \delta_{x,y} \qquad \forall\, x,y \in \mathbb{Z}^\nu \tag{2}$$

In the following, h will denote a hypercube $\{-L,\dots,0,\dots,L\}^\nu$ enclosing $|\Lambda| = (2L+1)^\nu$ points, and $\phi(x)$ are classical "spin" variables ranging over $\mathbb{R}$. The Hamiltonian of the spherical model in a random external field as described above is deflned by

$$H_\Lambda(\phi) = (\phi, -\Delta/2\ \phi) - v(\$,\$) - H \sum_{x \in \Lambda} \phi(x) - (h,\phi) \tag{3}$$

where the lattice Laplacean is given by

$$(-\Delta\phi)(x) = 2\nu\phi(x) - \sum_{i=1}^{\nu} \left[\phi(x+e_i) + \phi(x-e_i)\right]$$

Here e_i, $i=1,\dots,\nu$ is the unit vector in the i-th direction, and the scalar product is defined by

$$(f,g) = \sum_{x \in \Lambda} \overline{f(x)}\, g(x)$$

We denote the Gibbs expectation value by

$$\langle F \rangle_\Lambda \equiv \frac{\displaystyle\int \prod_{x \in \Lambda} d\phi(x)\ F(\phi)\ e^{-\beta H_\Lambda(\phi)}}{Z_\Lambda(\beta,\mu,H,h)} \tag{4}$$

where

$$Z_\Lambda(\beta,\mu,H,h) \equiv \int \prod_{x \in \Lambda} d\phi(x) \quad e^{-\beta H_\Lambda(\phi)} \tag{5}$$

is the partition function. The quenched free energy for the model is

$$f_\Lambda(\beta,\mu,H,h) \equiv - \frac{\beta^{-1}}{|\Lambda|} \langle \log Z_\Lambda(\beta,\mu,H,h) \rangle_\rho \tag{6}$$

together with the mean-spherical constraint

$$\langle\langle(\phi,\phi)\rangle_\Lambda\rangle_\rho = |\Lambda| \tag{7}$$

The above constraint may be expressed in terms of the chemical potential μ by

$$\left(\frac{\partial f_\Lambda(\beta,\mu,H,h)}{\partial\mu}\right)_{\beta,H,h} = -1 \tag{8}$$

We define the Fourier transformation $\hat{f}$ of a function $f: \Lambda \to C$ by

$$\hat{f}(k) = \frac{1}{\sqrt{|\Lambda|}} \sum_{x \in \Lambda} e^{-ikx} f(x) \quad ; \quad k \in \Lambda^* = \left\{\frac{x\,\pi}{2L+1} , x \in \Lambda\right\}$$

It is convenient to write

$$\hat{\phi}(k) = \begin{cases} \frac{1}{\sqrt{2}} \left[x(k) + iy(k)\right] & \text{if } k \neq 0 \\ x(0) & \text{if } k = 0 \end{cases}$$

and

$$\hat{h}(k) = \begin{cases} \frac{1}{\sqrt{2}} \left[\Gamma(k) + is(k)\right] & \text{if } k \neq 0 \\ \Gamma(0) & \text{if } k = 0 \end{cases}$$

Finally, the "spin wave energy" is given by

$$\varepsilon(k) = \sum_{i=1}^{\nu} (1 - \cos k_i) \quad , \quad k \equiv (k_i)_{i=1}^{\nu}$$

Upon performing the Gaussian integrals in eq.(5) we obtain

$$f_\Lambda(\beta,\mu,H,h) = \langle -(H+\Gamma_{0\Lambda})^2/4(\varepsilon(0)-\mu)$$

$$- \frac{1}{4|\Lambda|} \sum_{\substack{k \in \Lambda^* \\ k \neq 0}} \frac{\Gamma(k)^2 + s(k)^2}{\varepsilon(k)-\mu} \rangle_\rho$$

$$- \frac{\beta^{-1}}{2|\Lambda|} \sum_{\substack{k \in \Lambda^* \\ k \neq 0}} \{\log \pi - \log[2\beta(\varepsilon(k)-\mu)]\} \tag{9}$$

where

$$\Gamma_{0\Lambda} \equiv \frac{1}{|\Lambda|} \sum_{x \in \Lambda} h(x)$$

The constraint equation (8) takes the form (using $\varepsilon(0) = 0$)

$$F_\Lambda(\beta,\mu = \mu_\Lambda(\beta,H),H) \equiv \frac{H^2}{4\mu_\Lambda(\beta,H)^2} + \frac{1}{4\mu_\Lambda(\beta,H)^2} \langle\Gamma_{0\Lambda}^2\rangle_\rho$$

$$+ \frac{\beta^{-1}}{2|\Lambda|} \sum_{\substack{k \neq 0 \\ k \in \Lambda^*}} \frac{1}{\varepsilon(k) - \mu_\Lambda(\beta,H)}$$

$$+ \frac{1}{4|\Lambda|} \sum_{\substack{k \neq 0 \\ k \in \Lambda^*}} \frac{\langle\Gamma(k)^2 + s(k)^2\rangle_\rho}{(\varepsilon(k) - \mu_\Lambda(\beta,H))^2} = 1 \tag{10}$$

By assumptions eqs. (1) and (2) we obtain

$$\langle\Gamma(k)^2 + s(k)^2\rangle_\rho = \sigma^2 \qquad \forall\, k \in \Lambda^* \tag{11}$$

$$\langle\Gamma_{0\Lambda}^2\rangle_\rho = \frac{\sigma^2}{|\Lambda|} \tag{12}$$

We have made explicit the dependence of μ on the variables Λ, 6, H. We now assume that $\nu \leqslant 4$. It may be easily verified that for some c independent of Λ (for $|\Lambda|$ sufficiently large) there holds

$$\frac{\partial F_\Lambda}{\partial \mu} \geqslant \frac{\beta^{-1}}{2|\Lambda|} \sum_{\substack{k \in \Lambda^* \\ k \neq 0}} \frac{1}{(\varepsilon(k)-\mu)^2} \geqslant c > 0$$

if $\mu < 0$.

Since the integral

$$\int_B \frac{d^\nu k}{(2\pi)^\nu} \quad \frac{1}{(\varepsilon(k) - \mu)^2}$$

(where B is the first Brillouin zone) diverges as $\mu \to 0_-$ if $\nu \leq 4$ it is easy to prove that equation (10) has a unique solution, $\mu_\Lambda^0(\beta,H)$. Furthermore, $f_\Lambda(\beta,\mu,H,h)$ converges uniformly for μ in compact subsets of $(-\infty,0)$, and H in compact subsets of $\mathbb{R}$ to a function $f(\beta,\mu_0(\beta,H),H,h)$ (which is readily obtained explicitly from eq. (9)), where $\mu_0(\beta,H) < 0$ is the unique solution of the equation

$$\frac{H^2}{4\mu(\beta,H)^2} + \frac{\beta^{-1}}{2(2\pi)^\nu}\int_B \frac{d^\nu k}{\varepsilon(k) - \mu} + \frac{\sigma^2}{4(2\pi)^\nu}\int_B \frac{d^\nu k}{(\varepsilon(k) - \mu(\beta,H))^2} = 1 \tag{13}$$

By the implicit function theorem, μ_0 is continuously differentiable in H and

$$\mu_0(\beta, 0) < 0 \tag{14}$$

It is also easy to show that $\mu_\Lambda^0(\beta,H) \xrightarrow[|\Lambda|\to\infty]{} \mu_0(\beta,H)$. The magnetization

$$m_\Lambda(\beta,H) = \left(\frac{\partial f_\Lambda(\beta,\mu,H,h)}{\partial H}\right)_{\beta,\mu = \mu_\Lambda^0(\beta,H),H} = -\frac{H}{4\mu_\Lambda^0(\beta,H)}$$

satisfies

$$\lim_{H\to 0}\lim_{|\Lambda|\to\infty} m_\Lambda(\beta,H) = \lim_{H\to 0}\left[-\frac{H}{4\mu_0(\beta,H)}\right] = 0$$

because of eq.(14).

The above shows also that a ferromagnetic phase transition occurs at some nonzero critical temperature if $\nu \geq 5$ and σ is sufficiently small, i.e., whenever

$$\frac{\sigma^2}{4(2\pi)^\nu}\int \frac{d^\nu k}{\varepsilon(k)^2} < 1$$

We should like to thank P.A. Vuillermot for calling our attention to reference 1 and S.G. Rosa Jr. for pointing out to us reference 10.

REFERENCES

1. H.G.Schuster, Phys. Lett. *60*A, 89 (1977).
2. J.L. van Hemmen and R.G. Palmer, J. Phys. A*12*, 563 (1979).
3. J.F. Perez, W.F. Wreszinski and J.L. van Hemmen, to appear in J.Stat. Phys.
4. L.A. Pasthur, J. Stat. Phys. 27, 119 (1982).
5. H.E. Staniey, Phys. Rev. 176, *868 (1968).*
6. '. Fröhlich, B. Simon and T. Spencer, Comm. math. Phys.
7. J. Fröhlich, R. Israel, E. Lieb and B. Simon, Comm. math. Phys.
8. Phys. Today, Luly 1983, pp. 17-19.
9. J.F. Perez, Rev. *Bras. Fís.* 10, *193 (1980).*
10. P. Lacour Gayet and G. Toulouse, J. Phys. 35, 425 (1974).

Resumo

Calculamos a energia livre do modelo esférico médio em um campo magnético aleatório e provamos a ausência de ferromagnetismo se a dimensão do espaço for $\nu \leq 4$. Provamos também a existência de transição se $\nu \geq 5$ e a variança do campo for suficientemente pequena.