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Solitonic Gravitational Waves

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Abstract de present a solution to the Einstein equations for gravitational waves which exhibits soliton characteristics.

The soliton concept plays an important rolein several branches of physics. In particular, in relativistic field theory these non-trivial classical solutions are important for calculating the functional path integral through the semiclassical method. Thus we can calculate quantum fluctuations about these solutions and generate a well definite approximation technique for computing the Green's functions. Soliton solutions are relevant because they can not be reached by standard perturbation theory and therefore can provide information about the theory outside the perturbative context. Non-perturbative properties, like a mechanism for quark confinement in QCD, are hoped to be found in this way[1].

Gravitational solitons may also be very useful. Classically, they may be associated to galaxies in cosmological models[2] or may trigger gravitational wave detectors. At the big bang soliton configurations might have a fundamental role in determining the evolution of the Universe. And, at the quantum level, once a consistent quantum gravity theory is Formulated (the only surviving candidates at present being the supergrãvity theories[3]), the solutions will hopefully help in the understanding of the non-perturbative aspects of the theory.

In order to generate soliton solutions from Einsteinequations the inverse scattering[4,5] and Backlund transformation[6] methods have been adapted to general relativity. Vacuum axially symmetric waves and stationary axially symmetric space-times have been studied under these methods and shown to possess solitons. These techniques have also been extended to the case when matter is present behaving as solitons[2] or not[5]. However, the new solutions so obtained are in general complicated

$$P^0 = \int_{-\infty}^{+\infty} \sqrt{-g}\ \theta^{00} du \qquad (4)$$

Because we are dealing with plane waves we can only require a finite energy (density) flux and not finite energy. Also, the solutions will be required to be localized in the u-direction only.

We now return to Eq. (2) to find out an exact solution. Since β is an arbitrary function we choose it as

$$(\beta')^2 = 1 - L^2 \qquad (5)$$

so that (2) becomes

$$L'' + L(1 - L^2) = 0 \qquad (6)$$

This is the well-known equation for the kink in the $\lambda\phi^4$ theory in two dimensions[14]. Its non-trivial solutions are called topological solitons, and are localized solutions carrying finite energy and a conserved topological charge. The solutions of Eq. (6) are

$$L = \pm 1 \qquad (7a)$$

$$L = \pm\tanh \zeta, \quad \zeta = \frac{1}{\sqrt{2}} (u-u_0) \qquad (7b)$$

where u_0 is a constant. Since L enters in (1) and (5) only as L^2 the signs are not important. (7a) is the trivial solution since β=constant and the space-time is flat. For the solution (7b) we can integrate (5) to find

$$\beta = \pm\sqrt{2}\ \text{arc tan}\ (\sinh \zeta) + \beta_0 \qquad (8)$$

where β_0 is a constant. Since β_0 appears always as a multiplicative constant in (1) it can be set to zero. The signs in (8) are also irrelevant since the solution with a given sign can be obtained from the other by the exchange of x and y. Then, the element of line (1) can be put in the form

$$ds^2 = dudv + \tanh^2\zeta\ \{\exp\left[(\sqrt{2}\arctan(\sinh\zeta)\right]dx^2 + $$
$$+ \exp\left[-\sqrt{2}\ \arctan(\sinh \zeta)\right]dy^2\} \qquad (9)$$

This metric is also asymptotically flat since for $u \to \pm\infty$, $L^2 \to 1$ and $\beta \to$ constant. There is no physical singularity as is well known[10], the singularity appearing at $u = u_0$ can be removed by a suitable coordinate transformation[15].

and not easy to handle.

In this paper we present a simple soliton solution for a plane gravitational wave. Although suggested in ref. (7) that there are solitons among the solutions of plane gravitational waves none was presented there. Also, in ref. (7), the number of arbitrary functions associated to the model is incorrectly given as two. Once a transformation of coordinates is performed to eliminate $g_{,,}$ the number of arbitrary functions reduces to one[8], as is well known[9,10,11].

We start with the metric[10]

$$ds^2 = dudv + L^2(e^{2\beta}dx^2 + e^{-2\beta}dy^2) \tag{1}$$

with $L = L(u)$ and $\beta = \beta(u)$, and the field equations are given by

$$L'' + L(\beta')^2 = 0 . \tag{2}$$

where the prime denotes differentiation with respect to u. Eq. (1) represents a plane wave propagating along the u-axis, with amplitude L and wave factor β. The properties of such waves have been extensively studied for the case of a "sandwich" wave[9], that is, a wave for which the amplitude is non-zero only for a finite range of u, elsewhere the space-time being flat. In that case β is a pulse of duration $2T$ satisfying $|\beta'| << 1/T$ tnroughout the pulse. To study solitons, however, a solution without any approximation is desirable and we will later on present an exact solution to Eq. (2).

Since we are not going to apply the soliton generating techniques mentioned earlier we have to define solitons in the gravitational context. In the inverse scattering method[4,5], for example, a solution is called solitonic if the associated scattering matrix has poles, the number of poles being the number of solitons. Here, we will adopt a broader definition[12]: any non-trivial solution which is confined to a finite region of space-time and which carries a finite energy will be considered a soliton. In order to apply the last criterium we use the definition of the conserved energy-momentum pseudo-tensor given by[11,13]

$$\Theta^{\mu\nu} = \frac{1}{16\pi}\frac{\partial^2}{\partial x^\lambda \partial x^k}\left[\sqrt{-g}\ (g^{\mu\nu}g^{\lambda k} - g^{\mu\lambda}g^{\nu k})\right] \tag{3}$$

Since the plane waves move only in one direction, gravitational energy can be localized[13] and we can define an energy flux as

The only non-vanishing component of the energy-momentum pseudo -tensor (3) is

$$\theta^{vv} = \frac{1}{4\pi} \operatorname{sech}^2\zeta \, (3 \tanh^2\zeta - 1) \tag{10}$$

and the energy flux (4) is finite and equal to $2\sqrt{2}/(15\pi)$. We can also show, from (10), that the linear momentum in the z-direction is also finite as we would expect.

We thus have a soliton, a solution located around u_0, which is asymptotically flat and with a finite energy flux. It would be interesting to generalize this solution to a multi-soliton solution. To this end we could apply the inverse scattering method[4,5], first finding out the seed solution for the soliton presented here and them applying the machinery to generate the multi-soliton solution. It would also be possible to find out classes of solitons by stablishing the conditions L and β would satisfy in order that the integral (4) would be finite and the solutions asymptotically flat. We hope to report on this elsewhere.

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and finally by the coordinate transformation $d\bar{z}=g_4(z)dz$ we have $ds^2 = = d\bar{z}dw+g_2(\bar{z})dx_2^2+g_3(\bar{z})dx_3^2$.

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15. All the invariants constructed with the Riemann tensor vanish.

Resumo

Apresentamos uma solução das equações de Einstein para ondas qravitacionais que exibe caracteristicas de soliton.