Exact Propagator for Darnped Time-Dependent Harmonic Oscillator

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Abstract   Theworkof Montroll inderiving the propagatorof *time-depen-*dent harmonic oscillator is generalized to obtain the propagatoroftime-dependent harmonic oscillator with constant damping term.

1. INTRODUCTION

From Feynman's formulation of nonrelativistic quantummechanics the prcpagator, probability amplitude for a particle togofromthepoint (x',t') to the point (x'',t''), can be expressed as[1,2,3]

$$K(x'',t'';x',t') = \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} \exp\{(i/h)\int_{t}^{t''} L(x,\dot{x},t)dt\} \, Dx(t) \qquad (1.1)$$

where $L(x,\dot{x},t)$ is the Lagrangian of the dynamical system considered and $Dx(t)$ indicates that the integral is over all paths with fixed endpoints (x',t') and (x'',t'').

For time-dependent harmonic oscillator, Montroll[4] first transforms the path integral (1.1) into the Gaussian integral

$$\int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} \exp\{i(y^{T}Ay + 2b^{T}y)\} \prod_{j=1}^{n} dy_{j} = (i\pi)^{n/2}(\det A)^{-1/2}\exp(-ib^{T}A^{-1}b)$$

$$(1.2)$$

multiplied bya functionof x', x'' and τ. Here we have defined $\tau = (t''-t')/n$ for later convenience. He then carriers out calculations as $\tau \to 0$ (or $n \to \infty$). His method has recently been applied for evaluating the propagator of time-dependent forced harmonic oscillator[5]. In the present work the same method has been generalized further to calculate the propagator of time-dependent harmonic oscillator with constant damping.

In Section 2, we are able to transform our path integral into the Gaussian integral (1.2) multiplied by a function of x', x'' and τ.

In Section 3, we show the details of calculation (and also in the Appendix) as $\tau \to 0$ and we write down the propagator in terms of $f(t)$ and $g(t)$, which are respectively the solutions of time-dependent harmonic oscillator with damping and with antidamping. Finally, we discuss the result in Section 4.

2. FORMULATION

For time-dependent harmonic oscillator with constant damping term, the equation of motion is

$$\ddot{x} + \gamma\dot{x} + \omega^2(t)x = 0 \qquad (2.1)$$

where $\omega(t)$ is a time-dependent frequency and y is a constant damping coefficient. Eq. (2.1) can be obtained from the Lagrangian[6]

$$L(x,\dot{x},t) = e^{\gamma t}\, m[\dot{x}^2 - \omega^2(t)x^2]/2 \qquad (2.2)$$

In spite of its interpretation difficulties in quantum mechanics[7,8], we are going to use (2.2) as our Lagrangian. Now the propagator defined by (1.1) can be written as

$$K(x'',t'';x',x') = \lim_{n \to \infty} \left[\prod_{j=1}^{n} (me^{\gamma t_j}/2\pi i\hbar\tau)^{1/2}\right] \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} \exp\left\{(i\tau/2\hbar)\right.$$

$$\times \left[m\tau^{-2}\sum_{j=1}^{n} e^{\gamma t_j}(x_j - x_{j-1})^2 - m\sum_{j=0}^{n-1} e^{\gamma t_j}\omega_j^2 x_j^2\right]\right\} \prod_{j=1}^{n-1} dx_j \qquad (2.3)$$

by Feynman's definition. The extra factor $\exp(\gamma t_j)$ is necessary for including dissipative effect. For later convenience we have set $\tau = (t''-t')/n$ and $r_j = r(t'+j\tau)$, $r' = r(t')$ and $r'' = r(t'')$ for any function $r(t)$. Now we let $y_j = \exp(\gamma t_j/2)(m/2\hbar\tau)^{1/2}x_j$, then (2.3) can be rewritten as

$$K(x'',t'';x',t') = \lim_{n \to \infty} (i\pi)^{-n/2}(me^{\gamma t''}/2\hbar\tau)^{1/2} \exp\left\{(i\tau/2\hbar)\left[m\tau^{-2}(e^{\gamma t'}x'^2\right.\right.$$

$$+ e^{\gamma t''}x''^2) - me^{\gamma t'}\omega'^2 x'^2\right]\right\} \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} \exp\left\{i\left[\sum_{j=1}^{n-1}(1+e^{\gamma\tau}\omega_j^2\tau^2)y_j^2\right.\right.$$

$$\left.\left. - 2\sum_{j=0}^{n-1} e^{\gamma\tau/2} y_j y_{j+1}\right]\right\} \prod_{j=1}^{n-1} dy_j \qquad (2.4)$$

since $dx_j = \exp(-\gamma t_j/2)(2\hbar\tau/m)^{1/2}dy_j$.

By comparing (1.2) and (2.4) we find that the matrix A is of the form

$$A = \begin{pmatrix} a, & -d & 0 & 0 \ldots & 0 & 0 & 0 & 0 \\ -d & a, & -d & 0 \ldots & 0 & 0 & 0 & 0 \\ 0 & -d & a, & -d \ldots & 0 & 0 & 0 & 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ 0 & 0 & 0 & 0 \ldots -d & a_{n-3} & -d & 0 \\ 0 & 0 & 0 & 0 \ldots & 0 & -d & a_{n-2} & -d \\ 0 & 0 & 0 & 0 \ldots & 0 & 0 & -d & a_{n-1} \end{pmatrix} \qquad (2.5)$$

with $a_j = 1 + \exp(\gamma\tau) - \omega_3^2\tau^2$ and $d = \exp(\gamma\tau/2)$. The column matrix b has the following elements;

$$b_1 = - y' \exp(\gamma\tau/2) = - c\tau^{-1/2} \exp(\gamma t_1/2) \, x^1 \qquad (2.6)$$

$$b_j = 0 \quad (j = 2, 3, \ldots, n-2)$$

and

$$b_{n-1} = - y'' \exp(\gamma\tau/2) = - c\tau^{-1/2} \exp\left[\gamma(t''+\tau)/2\right]x''$$

Here we have set $c = (m/2\hbar)^{1/2}$. By substituting (1.2) into (2.4) we obtain

$$K(x'',t'';x',t') = \lim_{\tau\to 0} \left(me^{\gamma t''}/2\pi i\hbar\tau \det A\right)^{1/2} \exp\{iB(x'',x',\tau)\} \qquad (2.7)$$

and

$$B(x'',x',\tau) = (m/2\hbar\tau)(\exp(\gamma t_1) \, x'^2 + \exp(\gamma t'') \, x''^2) - b^T A^{-1}b \qquad (2.8)$$

We have assumed that the factor $\exp(-im \exp(yt') \omega'^2 x'^2\tau/2\hbar)$ in (2.4) to be one as $\tau \to 0$. Now we are only left to calculate the limit values of $\tau \det A$ and $B(x'',x',\tau)$ as $\tau \to 0$. With the help of (2.5) - (2.8), the calculations will be carried out in the next section and in the Appendix.

3. CALCULATION

From the matrix A we define A_3 and D_3 as the following **deter-**minants

$$A_1 = a_1, \quad A_2 = \begin{vmatrix} a_1 & -d \\ -d & a_2 \end{vmatrix} \quad, \quad A_3 = \begin{vmatrix} a_1 & -d & 0 \\ -d & a_2 & -d \\ 0 & -d & a_3 \end{vmatrix} \quad, \ldots, \quad A_{n-1} = \det A$$

$$D_{n-1} = a_{n-1} \,, \quad D_{n-2} = \begin{vmatrix} a_{n-2} & -d \\ -d & a_{n-1} \end{vmatrix}$$

$$D_{n-3} = \begin{vmatrix} a_{n-3} & -d & 0 \\ -d & a_{n-2} & -d \\ 0 & -d & a_{n-1} \end{vmatrix} \,, \ldots, \quad D_1 = \det A$$

It is easily to show that A_3 and D_j satisfy the recurrence relations

$$A_{j+1} = a_{j+1} A_j - d^2 A_{j-1} \,, \quad A_i = 1 \quad (1 \leqslant j \leqslant n-2) \tag{3.1}$$

and

$$D_{j-1} = a_{j-1} D_j - d^2 D_{j+1} \,, \quad D_n = 1 \quad (2 \leqslant j \leqslant n-1) \tag{3.2}$$

Furthermore, eqs (3.1) and (3.2) can be transformed into the finite--difference equations

$$(D_{j+1} - 2D_j + D_{j-1})/\tau^2 = -\omega_{j-1}^2 D_j - \gamma(D_{j+1} - D_j)/\tau \tag{3.3}$$

and

$$(A_{j+1} - 2A_3 + A_{j-1})/\tau^2 = -\omega_{j+1}^2 A_3 + \gamma(A_j - A_{j-1})/\tau \tag{3.4}$$

respectively. From eqs (3.1) and (3.2) we see that the end conditions of A_3 and D_3 are

$$D_{n-1} = a_{n-1} \approx 1 + 0(\tau) \approx a_i = A_i \tag{3.5}$$

$$(D_{n-1} - D_{n-2})/\tau = \{a_{n-1}(1 - a_{n-2}) + d^2\}/\tau \approx -(1/\tau) \tag{3.6}$$

and

$$(A_2 - A_1)/\tau = \{a_1(a_2 - 1) - d^2/\tau\} \approx 1/\tau \tag{3.7}$$

for small τ. In order to overcome the difficultieç of **divergence** in eqs (3.6) and (3.7), we now introduce f_j and g_j by

$$f_3 = \tau D_j \quad \text{and} \quad g_j = \text{TA}_j \qquad (3.8)$$

With the help of eqs (3.5) - (3.7), eqs (3.3) and (3.4) can berewritten as the differential equations

$$\mathbf{f} + \gamma\dot{f} + \omega^2(t)f = 0 \ , \ f'' = 0 \ , \ \dot{f}'' = -1 \qquad (3.9)$$

and

$$\ddot{g} - \gamma\dot{g} + \omega^2(t)g = 0 \ , \ g' = 0 \ , \ \dot{g}'' = 1 \qquad (3.10)$$

in the limit as $\tau \rightarrow 0$. Therefore, we obtain

$$\text{Lim}_{\tau\to 0} (\tau \det A) = \text{Lim}_{\tau\to 0} (\tau D_1) = \text{Lim}_{\tau\to 0} f_1 = f(t') = f' = g'' \qquad (3.11)$$

From eqs (3.1), (3.2) and (3.8) we discover that the f_3 and g_j are related through the formula

$$f_{j+1}g_j - d^2 f_{j+2}g_{j-1} = f_j g_{j-1} - d^2 f_{j+1}g_{j-2} = \tau^2 \det A = \tau f_1 = \tau g_{n-1}$$

$$(3.12)$$

Hence

$$g_j = \tau f_1 f_{j+2} (f_{j+1}f_{j+2})^{-1} + d^2 f_{j+2}g_{j-1}/f_{j+1}$$

$$= \tau f_1 f_{j+2} \{(f_{j+1}f_{j+2})^{-1} + d^2(f_j f_{j+1})^{-1}\} + d^4 f_{j+1}g_{j-2}/f_j$$

$$= \ldots = \tau f_1 f_{j+2} \sum_{k=1}^{j+1} (f_k f_{k+1})^{-1} d^{2(j-k+1)} \qquad (3.13)$$

In particular, we have

$$\tau^2 \sum_{k=1}^{n-1} (f_k f_{k+1})^{-1} d^{2(n-k)} = d^2 g_{n-2}/g_{n-1} \qquad (3.14)$$

for $j = n-2$. Using eqs (2.5) and (3.13) we can show that

$$b^T A^{-1} b = \sum_{j=1}^{n-1} (f_j f_{j+1}d^{2j})^{-1} \sum_{k=j}^{n-1} b_k f_{k+1}d^k)^2 \qquad (3.15)$$

The above relation is proved in the Appendix.

Now substituting eq (2.6) into eq (3.15), we get

$$b^T A^{-1} b = (f_1 f_2 d^2)^{-1} \{-c\tau^{-1/2} \left[e^{\gamma t_1/2} f_2 dx' + e^{\gamma(t''+\tau)/2} f_n d^{n-1} x'' \right] \}^2$$

$$+ \sum_{j=2}^{n-1} (f_j f_{j+1} d^{2j})^{-1} \{-c\tau^{-1/2} e^{\gamma(t''+\tau)/2} f_n d^{n-1} x''\}^2$$

$$= (c^2 f_2/\tau f_1) e^{\gamma t_1} x'^2 + (2c^2/f_1) e^{\gamma t''} x' x''$$

$$+ c^2 \tau \left[\sum_{k=1}^{n-1} (f_k f_{k+1})^{-1} d^{2(n-k)} \right] e^{\gamma t''} x''^2 \qquad (3.16)$$

after lengthy but straightforward calculations. Substituting eq (3.16) into eqs (2.7) and (2.8), the propagator takes the following form

$$K(x'',t'';x',t') = \underset{\tau \to 0}{\text{Lim}}\ (me^{\gamma t''}/2\pi i \hbar \tau \det A)^{1/2} \exp\{i \left[a_\tau x'^2 + b_\tau x' x'' + c_\tau x''^2 \right]\}$$

$$(3.17)$$

where

$$a_\tau = (me^{\gamma t_1}/2\hbar\tau)(1-f_2/f_1)$$

$$b_\tau = - me^{\gamma t''}/\hbar f_1$$

and

$$c_\tau = (me^{\gamma t''}/2\hbar\tau) \left[1 - \tau^2 \sum_{k=1}^{n-1} (f_k f_{k+1})^{-1} d^{2(n-k)} \right]$$

As $\tau \to 0$, we finally obtain

$$\underset{\tau \to 0}{\text{Lim}}\ a_\tau = \underset{\tau \to 0}{\text{Lim}}\ (me^{\gamma t_1}/2\hbar f_1)(f_1 - f_2)/\tau = - (me^{\gamma t_1} f'/2\hbar f') \quad (3.18)$$

$$\underset{\tau \to 0}{\text{Lim}}\ b_\tau = \underset{\tau \to 0}{\text{Lim}}\ (-me^{\gamma t''}/\hbar f_1) = - (me^{\gamma t''}/\hbar f') \qquad (3.19)$$

and

$$\underset{\tau \to 0}{\text{Lim}}\ c_\tau = \underset{\tau \to 0}{\text{Lim}}\ (me^{\gamma t''}/2\hbar\tau)(i - e^{\gamma \tau} g_{n-2}/g_{n-1})$$

$$= (me^{\gamma t''}/2\hbar)(-\gamma + \dot{g}''/g'') \qquad (3.20)$$

Here we have used eq (3.14) to derive eq (3.20).

Substituting eq (3.11) and (3.18) – (3.20) into eq (3.17), we obtain our main result

$$K(x'',t'';x',t') = (me^{\gamma t''}/2\pi i\hbar f')^{1/2} \exp\{(m/2i\hbar f')[\dot{f}'e^{\gamma t'}x'^2$$

$$+ 2e^{\gamma t''}x'x'' + (\gamma f' - \dot{g}'')e^{\gamma t''}x''^2]\} \tag{3.21}$$

Here we assume that $f' = g'' \neq 0$ for excluding catastrophic phenomenon. The propagator has been written in terms of $f(t)$ and $g(t)$ which are the solutions of the equation of motion of time-dependent harmonic oscillator with damping term and with antidamping term, respectively. For $\gamma=0$, the above equation is equivalent to eq (3.13) with $q(t) = 0$ in Ref. 5 as we expect.

4. RESULT

It can easily be shown that the solutions of eqs (3.9) and (3.10) are

$$f(t) = s(t)e^{-\gamma(t-t'')/2} \sin[\nu''-\nu(t)] \tag{4.1}$$

and

$$g(t) = s(t)e^{-\gamma(t'-t)/2} \sin[\nu(t) - \nu'] \tag{4.2}$$

respectively, where $s(t)$ and $\nu(t)$ are the amplitude and the phase of time-dependent harmonic oscillator with constant damping (or antidamping). In order to satisfy their boundary conditions, we let

$$\ddot{s}(t) - s'^2 s^{-3}(t) + \Omega^2(t)s(t) = 0 \; , \; \Omega^2(t) = \omega^2(t) - y^2/4 \tag{4.3}$$

and

$$s^2(t)\dot{\nu}(t) = s' \tag{4.4}$$

We alço have $s' = s''$, $\nu' = \dot{\nu}''$ and $s'\dot{\nu}' = s''\dot{\nu}'' = 1$ since $f' = g''$. With the help of eqs (4.1) - (4.4), eq (3.21) can be rewritten as

$$K(x'',t'';x',t') = [me^{\gamma(t'+t'')/2}\dot{\nu}'/2\pi i\hbar \sin \Phi(t'',t')]^{1/2} \times$$

$$\exp\{(m\dot{\nu}'/4i\hbar)[(2\dot{s}' - \gamma s')e^{\gamma t'}x'^2 - (2\dot{s}'' - \gamma s'')e^{\gamma t''}x''^2]\} \times$$

$$\exp \{(im\dot{\nu}'/2\hbar)[(e^{\gamma t'}x'^2 + e^{\gamma t''}x''^2)\cot \Phi(t'',t')$$

$$- 2e^{\gamma(t'+t'')/2} x'x'' \csc \Phi(t'',t')]\} \tag{4.5}$$

with $\Phi(\alpha,\beta) = \nu(\alpha) - \nu(\beta)$ for any two arbitrary time a and β. When $\omega(t)$ is a constant frequency ω_0, eq (4.5) reduces to the propagator evaluated by Papadopoulos[9] and is equivalent to (80) without perturbative force of Khandekar and Lawande[10]. However, we need both $f(t)$ and $g(t)$ to evaluate the propagator. It seems to agree with the idea of Feshbach and Tikochinsky[11].

APPENDIX

The elements of A^{-1}, represented by a_{jk}^{-1}, are determined by finding the cofactor of A. Therefore, we have from eq (2.5) that

$$a_{jk}^{-1} = d^{j-k} A_{k-1} D_{j+1}/D_1 = d^{j-k} g_{k-1} f_{j+1}/\tau f_1 , \qquad j > k \qquad (A.1)$$

and

$$a_{jk}^{-1} = d^{k-j} A_{j-1} D_{k+1}/D_1 = d^{k-j} g_{j-1} f_{k+1}/\tau f_1 , \qquad j \leqslant k \qquad (A.2)$$

With the help of eqs (3.13) - (3.14) and (A.1) - (A.2), we obtain

$$b^T A^{-1} b = \sum_{j,k=1}^{n-1} b_j a_{jk}^{-1} b_k$$

$$= (1/\tau f_1) \left\{ \sum_{k=1}^{n-2} b_k g_{k-1} d^{-k} \sum_{j=k+1}^{n-1} b_j f_{k+1} d^j + \sum_{j=1}^{n-1} b_j g_{j-1} d^{-j} \sum_{k=j}^{n-1} b_k f_{k+1} d^k \right\}$$

$$= \sum_{k=1}^{n-2} b_k f_{k+1} d^{-k} \left\{ \sum_{m=1}^{k} (f_m f_{m+1})^{-1} d^{2(k-m)} \right\} \sum_{j=k+1}^{n-1} b_j f_{j+1} d^j$$

$$+ \sum_{j=1}^{n-1} b_j f_{j+1} d^{-j} \left\{ \sum_{m=1}^{j} (f_m f_{m+1})^{-1} d^{2(j-m)} \right\} \sum_{k=j}^{n-1} b_k f_{k+1} d^k$$

$$= \sum_{j=1}^{n-1} (f_j f_{j+1} d^{2j})^{-1} \sum_{m=j}^{n-2} \sum_{k=m+1}^{n-1} (b_m f_{m+1} d^m)(b_k f_{k+1} d^k)$$

$$+ \sum_{j=1}^{n-1} (f_j f_{j+1} d^{2j})^{-1} \sum_{m=j}^{n-1} \sum_{k=m}^{n-1} (b_m f_{m+1} d^m)(b_k f_{k+1} d^k)$$

$$= \sum_{j=1}^{n-1} (f_j f_{j+1} d^{2j})^{-1} (\sum_{k=j}^{n-1} b_k f_{k+1} d^k)^2$$

after lengthy but straightforward calculatlons.

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Resumo

O trabalho de Montroll para deduzir o propagador do oscilador harmônico dependente do tempo é generalizado para obter o propagador do oscilador harmônico amortecido também dependente do tempo.