Revista Brasileira de Física, Volume 13, nº 2, 1983

Interacting Collective Modes in an Optical Cavity

E. L. GRAÇA and B. BASEIA*

Departamento de Física, Universidade Federal da Paraíba, 58000, João Pessoa, PB, Brasil

Recebido em 17 de fevereiro de 1983

Abstract We define collective modes for the quantized radiation field in a one-dimensional optical cavity coupled to a semi-infinite outside region and consider the interactions between the neighbouring collective modes to show how these interactions modify the time evolution of the free radíation field and investigate the limitations that they introduce on the exponential damping law. The procedure and results are preliminary towards the investigation of the single mode laser operation and Bose "condensation" aspect of laser behaviour near threshold.

1. INTRODUCTION

In a previous work[1], some fundamental aspects of the laser theory, such as the line narrowing mechanism and the fluctuation-dissipation theorem, were investigated via a continuous spectrum of modes generated by a semi-infinite one-dimensional optical cavity. This treatment is based on a model of cavity[2] coupled to the outside region and has the advantage of including the leakage of radiation field from the optical cavity in a natural way, instead of the usual phenomelogical loss introduced through a ficticious reservoir[3].

However, in the mentioned work[1], we analysed the transient and stationary solutions for the radiation field in the single mode operation. This approximation implies that we are neglecting the interactions among the collective modes generated by the optical cavity, due to the overlap between the neighbouring cavity bands. This is a reasonable approximation whenever the coupling between the optical cavity and the outside region is weak.

In the present work, we take into account the overlaps between the collective modes and investigate some modification they introduce in the solutions found in previous work. This treatment also allows one to get an improved insight into the question of Bose "condensation" aspect of laser behaviour near threshold. This paper deals with the free radiation field. In a future paper, we will include

* Partially supported by CNPq.

the active atoms inside the optical cavity in order to treat the complete problem.

2. MODEL AND FIELD MODES

In order to simulate a continuous spectrum of field modes we use a cavity model as a free space region bounded by two plane parallel plates, ona of which is ideally reflecting, placed at $z=\ell$, whereas the other one is semitransparent, placed at $z=0$, both perpendicular to the z - axis. We call $z \in [0,\ell]$ the interna1 region and $z \in (-\infty,0]$, the external one.

We take the plate coating the semitransparent window as a dielectric film which is modelled as a limiting case of a very thin layer with a very large dielectric constant, given by

$$\varepsilon(z) = \varepsilon_0 \left[1 + \eta\, \delta(z)\right] \tag{1}$$

where $\delta(z)$ is the Dirac delta function and η is a real parameter with dimensions of length, which determines the transparency of the window.

The normal modes of propagation are stationary solutions of Maxwell's equations that satisfy the boundary conditions. By assuming the electric field linearly polarized in x-direction: $\vec{E}(z,t) = E(z,t)\,\hat{x}$ and making the usual anzats for the Fourier field components

$$E_k(z,t) = U_k(z)\, \exp(-i\omega_k t) \tag{2}$$

we obtain, after some straightforward calculation, the normal modes for the entire cavity[1]

$$U_k(z) = \begin{cases} U_k^{(i)}(z) = M_k(n) \sin\, k_n(z-\ell) & , z \in [0,\ell] \\ U_k^{(e)}(z) = (2/\pi)^{1/2}\, \sin(kz-\delta_k), & z \in (-\infty,0] \end{cases} \tag{3}$$

where, for the case of a low-transmiting window, $M_k^2(n)$ is the Lorentzian lineshape

$$M_k^2(n) = (2/\pi)\, \Gamma_n^2\, \Lambda_{on}^2 / \left[\Gamma_n^2 + (\omega_{kn}-\omega_{on})^2\right] \tag{4}$$

and Γ_n is the linewidth given by[1]

$$\Gamma_n = c\ell/(\pi n\eta)^2 = c/\Lambda_{on}^2\ell \tag{5}$$

where $\Lambda_{0n} = \eta k_n = \eta n\pi/\ell$. ω_{0n} is the n-th Fox-Li quasimode[4]

$$\omega_{0n} = c\, k_n \cong cn\pi/\ell = c\,\Lambda_{0n}/\eta \qquad (6)$$

δ_k is a phase shift[1], and c is the speed of light. The case of a low-transmiting window results when we assume that the bandwidth Γ_n is small, in the sense that

$$\Gamma_n << \Delta\omega = |\omega_{0n} - \omega_{0,n\pm 1}| = c\pi/\ell \qquad (7)$$

The above results, such as those contained in eqs. (3)-(6), are reasonable approximations when the transmission throughout the window is so small that we can neglect the overlaps between the Lorentzians $M_k(n)$, $M_k(n')$. In fact, that was the case refered in our work andthat was the situation in which we assumed the single-mode operation.

Now, we consider the case in which the transmission is small - but not as assumed in the refered work - in such a way that the resulting lineshape of the cavity modes becomes no more a set of non-overlaping Lorentzian lineshapes, as given by eq. (3). So, as a consequence, we must substitute the Lorentzians $M_k(n)$ by another lineshape function L_k peaked around the Fox-Li quasimodes frequencies[1].

In this paper, we make a less restricted approximation: we assume that the resulting lineshape L_k can be described as a superposition of overlaping Lorentzians $\{M_k(n)\}$. We apply this procedure in the following section.

3. COLLECTIVE OPERATORS FOR THE RADIATION FIELD INSIDE THE CAVITY

By using the normal field modes (eq.(3)) and following the usual quantization procedure, the Hamiltonian for the free radiation field is (neglecting zero point energy)

$$H = \int_0^\infty \omega_k\, a_k^+ a_k\, dk \qquad (8)$$

where a_k^+ (a_k) creates (annihilates) photons with momentum k_n in the entire cavity $z \in (-\infty,\ell]$, with the usual canonical comrnutation relations

$$[a_k, a_{k'}] = [a_k^+, a_{k'}^+] = 0$$
$$[a_k, a_{k'}^+] = \delta(k-k') \tag{9}$$

The electric field operator, as expressed in terms of the operators a_k and a_k^+ is

$$E(z,t) = \int_0^m E_{0k} \, (a_k^+ + a_k) U_k(z) dk \tag{10}$$

where $E_{0k} = (\omega_k/2\varepsilon_0)^{1/2}$, $\omega_k = ck$.

In order to define the collective operator for the radiation field inside the cavity we refer to the electric field given by eq.(10) in the region $z \in [0,\ell]$

$$\begin{aligned} E^{(i)}(z,t) &= \int_0^\infty E_{0k} \, (a_k^+ + a_k) \, U_k^{(i)}(z) dk \\ &= \int_0^\infty E_{0k} \, (a_k^+ + a_k) \, L_k \sin k(z-\ell) dk \end{aligned} \tag{11}$$

as

$$\begin{aligned} E^{(i)}(z,t) &\simeq \sum_n \int_{B_n} E_{0k}(a_k^+ + a_k) M_k(n) \sin k(z-\ell) dk \\ &\simeq \sum_n E_{0k_n} \sin k_n(z-\ell) \int_{B_n} M_k(n) \, (a_k^+ + a_k) dk \end{aligned} \tag{12}$$

where the approximation

$$E_{0k} \sin k(z-\ell) M_k(n) \simeq E_{0k_n} \sin k_n(z-\ell) M_k(n) \tag{13}$$

has been used, since $E_{0k} \sin k(z-\ell)$ is a slowly varying function when compared to the Lorentzian function. So, instead of performing the integral as in eq. (11) we integrate in a band B_n, which has a Lorentzian profile given by $M_k(n)$, and afterwards sum over all the bands. In each band B_n we consider k running in the domain $k \in [0,\infty)$.

We now consider the internal electric field operator $E^{(i)}(z,t)$ which, according to eqs. (3) and (11), can be written in the form

$$E^{(i)}(z,t) = \sum_n M E_{0k_n} \sin k_n(z-\ell) \, (A_n^+ + A_n) \tag{14}$$

where M is a normalization factor, and

$$A_n = \frac{1}{M}\int_0^\infty M_k(n)\ a_k\ dk$$
$$A_n^+ = \frac{1}{M}\int_0^\infty M_k(n)\ a_k^+\ dk \qquad (15)$$

are the collective operators: $A_n^+(A_n)$ creates (annihilates) photons in the band (collective mode) B_n specified by the lineshape factor $M_k(n)$.

At this point we consider the (small) overlaps between the neighbouring bands in the following way: by using eqs. (9) it iseasy to show that

$$[A_n, A_n] = [A_n^+, A_n^+] = 0 \qquad (16)$$

and

$$[A_n, A_n^+] = \frac{1}{M^2}\int_0^m M_k(n)M_k(n')dk \qquad (17)$$

So, if we neglect the overlap between the Lorentzians $M_k(n), M_k(n')$, then

$$M_k(n)M_k(n') = 0, \quad n' \neq n \qquad (18)$$

and setting the normalization constant

$$M = \left(\int_0^\infty M_k^2(n)\ dk\right)^{+1/2} \qquad (19)$$

we obtain

$$[A_n, A_{n'}^+] = \delta_{n,n'} \qquad (20)$$

However, if we consider the mentioned overlap, we must calculate the integral in eq. (17). In order to do that, we use the Fourier transforms of $M_k(n)$, $M_k(n')$ and obtain, after some algebra (Appendix)

$$[A_n, A_{n'}^+] = P_{-1/2}\left[(s^2+\beta^2+1)/2\right] \qquad (21)$$

where $P_{-1/2}(X)$ is the toroidal function[5] and

$$s = (\omega_{0n} - \omega_{0n'})\ /\ \Gamma_n$$
$$\beta = \Gamma_n/\Gamma_{n'} = (n'/n)^2 \simeq 1 \qquad (22)$$

since, according to eq. (5)

(23)

is a slowly-varying function of the ressonance frequency within the optical dornain, where $n \sim 10^6$.

Using eqs. (21) and (22) we obtain for $n' = n \to s = 0$

$$[A_n, A^+_{n'}] = P_{-1/2}(1) = 1 \tag{24}$$

But for $n' \neq n$ this implies, according to eqs. (7) and (22) $s=\Delta\omega/\Gamma_n >> 1$. This leads to the toroidal function $P_{-1/2}(x)$ in the assymptotical limit. So, a little algebrical procedure gives for $n' = n\pm1$

$$[A_n, A^+_{n'}] = P_{-1/2}\ (s^2/2) \to 2/s \tag{25}$$

The result $2/s$ appearing in the right hand-side of eq. (25) measures the degree of non-orthogonality between two neighbouring collective modes. In the next section we use eqs. (24) and (25) in order to investigate the influence of the interactions between collective modes on the free radiation field.

4. TIME EVOLUTION OF THE FREE FIELD INSIDE THE CAVITY

We consider, for simplicity, the case of only two collective modes in the optical cavity, and use the density operator[3] $\rho(t)$ to describe the radiation field inside the cavity. So, in terms of the collective operators and adopting the antinormal order[3,6] we set

$$\rho(t) = \sum_{m,m',\ell,\ell'} C_{mm'\ell\ell'}(t) A^m_n A^{m'}_{n'} (A^+_n)^\ell (A^+_{n'})^{\ell'} \tag{26}$$

The equation of motion for the density operator is[7,8] ($\hbar = 1$)

$$\frac{d\rho}{dt} = i L \rho + L\rho \tag{27}$$

where L is the effective Liouville operator

$$L\rho = [H_0, \rho] \tag{28}$$

where H_0 is the effective Hamiltonian

$$H_0 = \sum_n \omega_n A^+_n A_n \tag{29}$$

and L is the loss operator

$$L\rho = \sum_j \Gamma_j \left[A_j, \rho A_j^+\right] + \text{h.c.} \tag{30}$$

where h.c. stands for the hermitian conjugate. The non-hamiltonian dynamic, as expressed by eq. (27), is a direct consequence of the fact that the system we are considering is an open system. It should be stressed that eq. (27) is valid if we neglect the interactions between the field modes. We correct it afterwards through eq. (33), in order to introduce these interactions.

Substituting eq. (28)-(30) in eq. (27) and making use of the interaction picture, where

$$\rho_I = e^{itH_0} \rho \, e^{-itH_0} \tag{31}$$

and omitting, for brevity, the index I, we obtain

$$\begin{aligned} d\rho/dt = \; & \Gamma_n \left[A_n^+ A_n \rho + \rho A_n^+ A_n - 2A_n \rho A_n^+\right] \\ & + \Gamma_{n'} \left[A_{n'}^+ A_{n'} \rho + \rho A_{n'}^+ A_{n'} - 2A_{n'} \rho A_{n'}^+\right] \end{aligned} \tag{32}$$

At this point we introduce the interaction between the collective modes into the equation of motion (eq. (27)) through the use of eq. (25): although ρ is in the antinormal order, the right hand-side of eq. (32) is not; in order to set it also in the antinormal order we use eq. (26) for ρ and the slight generalized identities

$$\left[A_n, A_{n'}^{+m}\right] = \begin{cases} \partial A_n^{+m}/\partial A_n^+ & ; \; n' = n \\ \\ (2/s)\,\partial A_n^{+m}/\partial A_n^+ & ; \; n' = n \pm 1 \end{cases} \tag{33}$$

and

$$\left[A_n^+, A_{n'}^m\right] = \begin{cases} -\,\partial A_n^m/\partial A_n & ; \; n' = n \\ \\ -(2/s)\,\partial A_n^m/\partial A_n & ; \; n' = n \pm 1 \end{cases} \tag{34}$$

where the eqs. (24) and (25) have been used. We obtain (put $\Gamma_n = \Gamma_{n'} = \Gamma$)

$$\frac{d\rho}{dt} = \Gamma \left[\frac{\partial (A_n\rho)}{\partial A_n} + \frac{\partial (\rho A_n^+)}{\partial A_n^+} + \frac{\partial (A_{n'}\rho)}{\partial A_{n'}} + \frac{\partial (\rho A_{n'}^+)}{\partial A_{n'}^+} + \frac{2}{s}\left(A_{n'}\frac{\partial\rho}{\partial A_n} + \frac{\partial\rho}{\partial A_n^+}A_{n'}^+ + A_n\frac{\partial\rho}{\partial A_{n'}} + \frac{\partial\rho}{\partial A_{n'}^+}A_n^+\right)\right] \tag{35}$$

The antinormal form, as in the foregoing equation, is appropriate for the use of the coherent states. So, we now make use of the coherent representation[9], where

$$\rho(t) = \frac{1}{\pi}\int |\{v_n\}\rangle \langle\{v_n\}|\; P\, d^2\{v_n\} \tag{36}$$

and $|\{v_n\}\rangle$ is a collective coherent state: $A_n|\{v_n\}\rangle = v_n|\{v_n\}\rangle$ and $P = P(\{v_n\}, \{v_n^*\}, t)$ is a quasiprobability function[9]. In this way, for the case of only two modes we obtain, after some algebrical manipulations

$$\frac{dP}{dt} = \Gamma\left[\frac{\partial(v_n P)}{\partial v_n} + \frac{\partial(v_n^* P)}{\partial v_n^*} + \frac{\partial(v_{n'}P)}{\partial v_{n'}} + \frac{\partial(v_{n'}^* P)}{\partial v_{n'}^*} + \frac{2}{s}\left(v_{n'}\frac{\partial P}{\partial v_n} + v_{n'}^*\frac{\partial P}{\partial v_n^*} + v_n\frac{\partial P}{\partial v_{n'}} + v_n^*\frac{\partial P}{\partial v_{n'}^*}\right)\right] \tag{37}$$

Setting

$$v_n = x + iy$$
$$v_{n'} = x' + iy' \tag{38}$$

the imaginary parts in the right hand-side of eq. (37) cancel, leading it to the form

$$\frac{dP}{dt} = \Gamma\left[\vec{\nabla}_{\vec{r}}.(\vec{r}P) + \vec{\nabla}_{\vec{r}'}.(\vec{r}'P) + \frac{2}{s}\left(\vec{r}.\vec{\nabla}_{\vec{r}'}P + \vec{r}'.\vec{\nabla}_{\vec{r}}P\right)\right] \tag{39}$$

where $\vec{r}(\vec{r}')$ is a vector with components $x,y(x',y')$. It is easy to see that if we neglect the overlaps between collective modes then the quasiprobability $P(\vec{r},\vec{r}',t)$ becomes separable: $P(\vec{r},\vec{r}',t) \to P_1(\vec{r},t).P_2(\vec{r}',t)$. In this particular case eq. (39) decouples in two independent equations and gives

$$\frac{dP_1}{dt} + \vec{\nabla}_{\vec{r}} (-\Gamma\vec{r}P_1) = 0 \tag{40}$$

which is a continuity equation whose solution is[10]

$$\vec{r}(t) = \vec{r}(0)\exp(-\Gamma t) \tag{41}$$

A similar equation is valid for $\vec{r}'(t)$. This sesult shows that, when the interaction between the collective modes is neglected, the time evolution of the free radiation field inside the cavity ($z \in [0,\ell]$) is of exponentlal-type. The inclusion of the interaction between these modes destroy this exponential damping behaviour. In order to show this we go back to eq. (39) and, assuming that the interaction is small, we set

$$P(\vec{r},\vec{r}',t) = P_1(\vec{r},\vec{r}',t)P_2(\vec{r},\vec{r}',t) \tag{42}$$

in such a way that

$$\begin{aligned} (1/P_1)\nabla_r P_1 &\gg (1/P_2)\nabla_r P_2 \\ (1/P_1)\nabla_{r'} P_1 &\ll (1/P_2)\nabla_{r'} P_2 \end{aligned} \tag{43}$$

Tnis means that we are assuming $P_1 (P_2)$ is a slowly-varying function of $\vec{r}'(\vec{r})$ when compared to $P_2(P_1)$. Sübstituting eq. (42) back in eq. (37) and using the approximation given in (43) we obtain the solutions

$$\vec{r}(t) = \vec{r}(0) \left[1 + (2/s^2)\Gamma^2 t^2\right] e^{-\Gamma t} \tag{44}$$

and

$$\vec{r}'(t) = \vec{r}(0) \left[(2/s)\Gamma t\right] e^{-\Gamma t} \tag{45}$$

where the initial condition $\vec{r}'(0) = 0$ has been used. This is the easest situation to investigate the influence of one mode on another, via the interaction. Also, it is the most favorable situation in ordertoobtain a solution corresponding to only one excited eollective mode. It iseasy to verify that if we neglect the mentioned interaction we recover in eq. (44) the result given by eq. (41).

The results contained in eqs. (44) and (45) allows us to derive the restriction to the exponential damping. In fact, according to eq.

(44), the exponential damping for $\vec{r}(t)$ is valid for times srnall enough such that

$$(2/s)\Gamma t << 1 \tag{46}$$

which gives

$$\frac{t}{\tau} << \frac{s}{2} = \frac{\Delta\omega}{2\Gamma} \simeq \frac{\tau}{\tau_0} \tag{47}$$

where $\tau_0 = \ell/c$, $\tau = \Gamma^{-1} = \Lambda_n^2\,\tau_0$. For a low-transmiting window that we are assuming throught this paper $\Lambda_n^2 \gg 1$ ($\Lambda_n^2 \gtrsim 10^3$ for transmission less than one percent) and it is possible to characterize the exponential damping for several life-times of the free radiation field in the internal cavity. For times other than those quoted in eq. (46) we seethat $\vec{r}'(t)$ cannot be neglected, since $\vec{r}'(t) \neq 0$ irrespective of the initial condition $\vec{r}'(0) = 0$, and the exponential damping behaviour ceases to be valid. This shcws that the interaction between the collective modes, due to the transmission throughout the window, introduces an upper bound to the exponential damping and, at the same time, forbides steady-stateexcitation into a single mode operation.

According to the foregoing results, whenever we try to excite the single-mode $\vec{r}(t)$, we also excite others neighbouring modes $\vec{r}'(t), \vec{r}''(t), \ldots$, etc. This is not a surprising result. However, the present treatment, in terms of a more realistic model of cavity yielding interacting collective modes, besides being a non-phenomenological procedure, enables one to get a more satisfactory approach of the Bose "condensation" aspect of the laser behaviour near (bellow) threshold.

We have also analyzed this question in the appropriate context of a many-mode theory of the present variety and will publish elsewhere.

APPENDIX

In this Appendix we derive eq. (21). To this end, we set

$$[A_n, A_{n'}^+] = \frac{1}{M^2}\int_0^\infty M_k(n)\, M_k(n')dk \tag{A1}$$

and take the Fourier transform of $M_k(n)$

$$\tilde{M}_y(n) = (\Lambda_n \sqrt{\pi}\,)\int_{-\infty}^{\infty} e^{-iuy}(1+u^2)^{-1/2}\, du \tag{A2}$$

where $u = (\omega_{kn} - \omega_{0n})/\Gamma_n$, and obtain

$$\tilde{M}_y(n) = (2\Lambda_n \sqrt{\pi})\ K_0(|y|) \tag{A3}$$

where $K_0(|y|)$ is the modified zero order Bessel function of second kind. Using[1]

$$M^2 = \int_0^\infty M_k^2(n)dk = \pi/c\Gamma_n \tag{A4}$$

and (A_2), (A_3) in (A_1) we find

$$[A_n, A_{n'}^+] = \int_0^\infty K_0(|y|)K_0(|y \frac{\Gamma_{n'}}{\Gamma_n}|)e^{iy(\frac{n}{\Gamma_n} - \frac{n'}{\Gamma_{n'}})} dy \tag{A5}$$

which[11] leads to eq. (21).

FOOTNOTES AND REFERENCES

1. B.Baseia and H.M.Nussenzveig, preprint, IFUSP/P-308, 1981.
2. R.Lang, M.O.Scully and W.E.Lamb, Jr., Phys. Rev. A7, 1788 (1973).
3. W.H.Louisell, *Quantum Statistical Properties of Radiation* (Willey-N.York, 1973).
4. See, for example, M.Sargent III, M.O.Scully and W.E. Lamb, Jr., *Laser Physics* (Addison-Wesley, London, 1974).
5. M.Abramowitz, *Handbook of Mathematical Functions* (Dover, 1968).
6. H.M. Nussenzveig, *Introduction to Quantum Optics* (Gordon and Breach, N.York, 1973).
7. R.Bonifacio and L.A.Lugiato, Phys. Rev. A*11*, 1507 (1975).
8. B.Baseia, Ph.D. Thesis, Universidade de São Paulo 1979 (unpublished).
9. R.J.Glauber, Phys. Rev. *130*, 2529 (1963); 232, 2766 (1963).
10. S.Chandrasekhar, Rev. Mod. Phys. 15, 34 (1943).
11. A.Ederly et *al.*, *Higher Transcendental Function* (Bateman Project, Mc.Graw-Hill Book Comp. N.York, 1953).

Resumo

São definidos modos coletivos para o campo de radiação quantizado em uma cavidade óptica unidimensional acoplada a uma região externa semi-infinita. Levamos em conta as interações entre os modos coleti-

vos vizinhos para mostrar como elas modificam a evolução temporal do campo de radiação livre e limitações que impõem ao decaimento exponencial. O procedimento e resultados são preliminares para a investigação da operação laser em "modo único", bem como o aspecto da "condensação" de Bose, que ocorre no laser próximo do limiar.