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On a Class of Exact Solutions of Poincaré Gauge Field Theory in Vacuum

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Abstract We present a detailed discussion of certain exact solutions, in vacuum, of our recently proposed equations for Poincaré gauge field theory. The construction of the solutions involves ansätze and methods well-known in Instanton Physics.

1. INTRODUCTION

Recently we proposed a dynamics of Poincaré gauge theory based upon a generalised Hilbert action in the principal bundle of orthogonal frames over space-time, endowed with a metric constructed from the Lorentz connection and the solder form[1]. In the absence of matter the only dynamical variables are:

a) the vierbein field $s^{i}{}_{\alpha}(x)$ and its inverse $a^{\alpha}{}_{i}(x)$, with the related metric $g_{ij}(x) = \eta_{\alpha\beta}\, \sigma^{\alpha}{}_{i}(x)\, \sigma^{\beta}{}_{j}(x)$ (*)

b) the metric connection coefficients $\Gamma_{i}^{[\alpha\beta]}(x)$, antisymmetric in $[\alpha,\beta]$ and given in matrix notation by Γ_{i} with matrix elements $\Gamma^{\alpha}_{i\beta} = \eta_{\beta\gamma}\,\Gamma_{i}^{[\alpha\gamma]}$.

The equations of motion are:

$$\nabla^{i} F_{ij}{}^{\alpha}{}_{\beta} = 0 \tag{1}$$

$$R_{ij} - \frac{1}{2} R\, g_{ij} + \Lambda\, g_{ij} + T_{ij} = 0 \tag{2}$$

where

$$F_{ij} = \partial_{i}\Gamma_{j} - \partial_{j}\Gamma_{i} + [\Gamma_{i},\Gamma_{j}]$$

is the curvature of the Lorentz connection, while R_{ij} and R are respectively the Ricci tensor and the curvature scalar of the Riemannian

(*): Latin indices refer to holonomic components, while the greek indices, single or in antisymmetric pairs, refer to the Lie algebra of the Poincaré group. The Minkowski metric is $\eta = \mathrm{diag}(+,-,-,-)$.

connection defined by the Christoffel symbols $\{{p \atop iq}\}$.

$$R_{k\ell}{}^{p}{}_{q} = \partial_k\{{p \atop \ell q}\} - \partial_\ell\{{p \atop kq}\} + \{{p \atop kr}\}\{{r \atop \ell q}\} - \{{p \atop \ell r}\}\{{r \atop kq}\}$$

$$R_{kq} = R_{k\ell}{}^{\ell}{}_{q} \quad \text{and} \quad R = g^{kq} R_{kq}$$

In eq (2) we allow for a cosmological constant A and T_{ij} is the "energy-momentum" of the field F_{ij} given by

$$T^{i}{}_{j} = \lambda(\mathrm{Tr}(F^{ik} F_{jk}) - \frac{1}{4}\delta^{i}{}_{j}\,\mathrm{Tr}(F^{k\ell} F_{k\ell}))$$

where h is a coupling constant with the dimensions of a length squared. The covariant derivative in eq (1) is that introduced by C.N.Yang[2]: the connection coefficients for holonornic indices are the Christoffel symbols, while for anholonomic indices they are given by the $\Gamma^{a}_{i\beta}$'s.

In a holonomic gauge the Lorentz connection and curvature are given by

$$\Gamma^{p}_{iq} = s^{p}{}_{\alpha}\,\Gamma^{\alpha}_{i\beta}\,\sigma^{\beta}{}_{q} + s^{p}{}_{\alpha}\,\partial_i\sigma^{\alpha}{}_{q}$$

$$F_{ij}{}^{p}{}_{q} = s^{p}{}_{\alpha}\,F_{ij}{}^{\alpha}{}_{\beta}\,\sigma^{\beta}{}_{q}$$

and eq (1) turns into

$$\nabla^{i}F_{ij}{}^{p}{}_{q} + [K_i, F_{ij}]^{p}{}_{q} = 0 \qquad (1)'$$

where

$$K^{p}_{iq} = \Gamma^{p}_{iq} - \{{p \atop iq}\}$$

is the contortion tensor.

In section 2 we introduce the ansätze leading to exact solutions of the coupled eqs (1)' and (2). The work of J. Cervero, L. Jacobs and C.R. Nohl[3] and of A. Actor[4] is briefly recalled in section 3. Their construction of solutions of the equation $\Box f/f^3$ = constant, in Minkowski space-time, is applied to our problem in section 4. Some final comments are made in section 5.

2. THE ANSÄTZE

Generalising the 't Hooft-Corrigan-Fairlie-Wilczek[5] ansatz to curved space-time, we look for solutions of eqs (1)' and (2) with torsion generated by a scalar function. The contortion tensor takes the form

$$K^{p}_{iq} = A_{ij\ q}^{\ \ p}\ \partial^{j} \ln \phi \tag{3}$$

with

$$A_{ij\ q}^{\ \ p} = \delta^{p}_{\ i}\ g_{jq} - \delta^{p}_{\ j}\ g_{iq}$$

The corresponding curvature is given by

$$F_{ij\ q}^{\ \ p} = R_{ij\ q}^{\ \ p} - \frac{1}{2} A_{ij\ q}^{\ \ p} \frac{\Box\phi}{\phi} \quad A_{jk\ q}^{\ \ p}\ S_{i}^{\ k}(\phi;g) - A_{ik\ q}^{\ \ p}\ S_{j}^{\ k}(\phi;g) \tag{4}$$

where

$$S_{i}^{\ k}(\phi;g) = \frac{\nabla_{i}\partial^{k}\phi}{\phi} - 2\,\frac{\partial_{i}\phi\ \partial^{k}\phi}{\phi^{2}}\ \frac{1}{4}\,\delta^{k}_{\ i}\left(\frac{\Box\phi}{\phi} - 2\,\frac{\partial\phi\cdot\partial\phi}{\phi^{2}}\right) \tag{5}$$

and eq (1)' becomes

$$\nabla^{i} W_{ij\ q}^{\ \ p} + W_{ij\ q}^{\ \ p} \frac{\partial^{i}\phi}{\phi} - \frac{1}{2} A_{ij\ q}^{\ \ p}\ \phi^{2}\ \partial^{i}\,\frac{\Box\phi + R/6\ \phi}{\phi^{3}} = 0$$

where $V_{ij\ 4}^{\ \ p}$ is the conforma1 Weyl tensor

$$W_{ij\ q}^{\ \ p} = R_{i\ j\ q}^{\ p} + \frac{1}{2}\left(A_{ik\ q}^{\ \ p}\ R^{k}_{\ j} - A_{jk\ q}^{\ \ p}\ R^{k}_{\ i}\right) - R/6\ A_{ij\ q}^{\ \ p}$$

A contraction of the indices j and p shows that the above equation is equivalent to the pair

$$\partial_{i}\ \frac{\Box\phi + R/6\ \phi}{\phi^{3}} = 0 \tag{6a}$$

$$\nabla^{i} W_{ij\ q}^{\ \ p} + W_{ij\ q}^{\ \ p} \frac{\partial^{i}\phi}{\phi} = 0 \tag{6b}$$

To simplify further we assume that space-time is conformally flat so that the Weyl tensor vanishes and eq (6b) is automatically satisfied. The metric tensor is locally of the form

$$g_{ij}(x) = \rho^{2}(x)\ \eta_{ij} \tag{7}$$

From the identity

$$\frac{\Box\phi + R/6\ \phi}{\phi^{3}} = \frac{\Box_{M}\psi}{\psi^{3}}$$

where $\psi = \rho\phi$ and $\Box_{M}$ is the Minkowski Dalembertian, it follows that eq (1)' finally reduces to

$$\Box_M \psi/\psi^3 = \mu \tag{8a}$$

where μ is a constant.

Since the energy-momentum T_{ij} is traceless, eq (2) implies that the curvature scalar is constant

$$R = 4\Lambda$$

For conformally flat metrlcs one has the identity

$$R/6 = \Box_M \rho/\rho^3$$

which now becomes an equation for the conformal factor p:

$$\Box_M \rho/\rho^3 = (2/3)\Lambda \tag{8b}$$

Finally with the above ansätze we notice that

$$R_{ij} - \frac{1}{4} R\, g_{ij} = 2\, S_{ij}(\rho;\eta)$$

and

$$T_{ij} = -4\lambda\phi^2 \frac{\Box_M\psi}{\psi^3} S_{ij}\ (\psi;\eta)$$

Substituting these results in eq (2) yields

$$\rho^2\, S_{ij}(\rho;\eta) - 2\lambda\mu\,\psi^2 S_{ij}(\psi;\eta) = 0 \tag{8c}$$

The ansätze expressed in eqs (3) and (7) have thus reduced the problem of solving the eqs (1)' and (2) to that of finding solutions of eqs (8a) and (8b) that are related by eq (8c).

3. THE MERONI SOLUTION AND THEIR ELLIPTIC EXTENSION*

It has been shown[3,4] that, if $f(x)$ is a solution of

$$\Box_M f/f^3 = \nu_0$$

then a solution $F(x) = f(x)\ E(u(x),m)$ of

$$\Box_M\ F/F^3 = \nu$$

can be constructed by the following procedure:

1) the Jacobian elliptic function $E(u,m)$ of argument u and parameter m is solution of the differential equation

$$E'' + a\,E + b\,E^3 = 0$$

* Squares and products of fourvectors in this section are to be taken with the Hinkowski metric.

where the constants a and b are functions of the parameter m, depending on the particular elliptic function considered[6].

2) the function $u(x)$ has to satisfy the equations

$$2\,\partial u.\partial f + f\,\Box_M u = 0 \tag{9}$$

and

$$a\,\partial u.\partial u = \nu_0\,f^2 \tag{10}$$

3) the constant v is related to v, by $\nu = -(b/a)\,v,$.

We shall be particularly interested in the meron function

$$f(x) = A((x+iv)^2(x-iv)^2)^{-1/2}$$

which is solution of

$$\Box_M f/f^3 = 4v^2/A^2$$

where v is an arbitrary spacelike or timelike fourvector.
The corresponding function $u(x)$ exists and is given by

$$u(x) = a^{-1/2}\,\theta(x)\;,$$

where

$$\theta(z) = (i/2)\ln\left[(x-iv)^2/(x+iv)^2\right] = \mathrm{tg}^{-1}\tau(z)$$

with

$$\tau(x) = 2x.v/(x^2-v^2)$$

The pair of functions $f(x)$ and $\theta(x)$ satisfy the remarkable properties:

$$f\,\partial_k\,\partial_l\theta = \partial_k\theta\,\partial_l f + \partial_l\theta\,\partial_k f - \eta_{kl}\,\partial\theta.\partial f \tag{9'}$$

$$\partial_k\theta\,\partial_l\theta = \frac{1}{4}\,\eta_{kl}\,\nu_0\,f^2 - S_{kl}(f;\eta) \tag{10'}$$

These properties imply eqs (9) and (10) and also that

$$F^2\,S_{kl}(F;\eta) = 2\,(c/a)f^2\,S_{kl}(f;\eta)$$

where c is the value of the first integral of the elliptic equation

$$c = E'^2 + a\,E^2 + (b/2)\,E^4$$

and is a function of the parameter m.

The singularities of $f(x)$ and the choice of the branch defining $\theta(x)$ depend on the type of the fourvector v. We consider 1) *Timelike* v ($v^0 > 0$, without loss of generality). The meron function $f(x)$ is regular everywhere in Minkowski space and the two-sheeted hyperboloid, $x^2-v^2=0$, divides Minkowski space in three regions:

$$I = \{x|\ x^2 - v^2 < 0\ \}$$
$$II_+ = \{x|\ x^2 - v^2 > 0,\ x.v > 0\}$$
$$II_- = \{x|\ x^2 - v^2 > 0,\ x.v < 0\}$$

We may define $\theta(x)$ continuously over the whole of Minkowski space by:

$$\theta(x) = \begin{cases} \text{Arctg}\ \tau(x) & , x \in I \\ \text{Arctg}\ \tau(x) - \pi & , x \in II_+ \\ \text{Arctg}\ \tau(x) + \pi & , x \in II_- \end{cases}$$

where Arcty is the principal branch of tg^{-1}. In this range $\theta(x)$ varies from $-\pi$ to $+\pi$, but if space-time is confined to the region π, thevalues of $\theta(x)$ are restricted to $[-\pi/2, +\pi/2]$.

2) *Spacelike v*

The one-sheeted hyperboloid, $x^2-v^2 = 0$, has now a non-void intersection with the plane $x.v = 0$ and $f(x)$ is singular there. At fixed time x^0 the singularity is located on a circle in the plane $x.v = 0$. We have now four regions to consider:

$$I_+ = \{x|\ x^2 - v^2 > 0,\ x.v > 0\}$$
$$I_- = \{x|\ x^2 - v^2 > 0,\ x.v < 0\}$$
$$II_+ = \{x|\ x^2 - v^2 < 0,\ x.v > 0\}$$
$$II_- = \{x|\ x^2 - v^2 < 0,\ x.v < 0\}$$

and also $I = I_+ \cup I_-$, $II = II_+ \cup II_-$.

To determine $\theta(x)$ we have two possibilities, one of which is

$$\Theta(x) = \begin{cases} \text{Arctg}\ \tau(x) & x \in I \\ \text{Arctg}\ \tau(x) + \pi, & x \in II_+ \\ \text{Arctg}\ \tau(x) - \pi, & x \in II_- \end{cases}$$

The function $\theta(x)$ has a discontinuity of 2π across the plane $x.v = 0$ in region II and is not defined on the two-dimensional hyperboloid $x^2-v^2=0$,

$x.v = 0$. Over the rest of Minkowski space $\theta(x)$ varies in the interval $[-\pi,+\pi]$. Agaln if we confine space-time to region I, $\theta(x)$ varies in the range $[-\pi/2, +\pi/2]$.

4. THE SOLUTIONS

A study of the twelve Jacobian elliptic functions[6] shows that there are only eigth independent solutions: a first class of four {sn, ns, cd, dc}, with argument $u(x) = (1+m)^{-1/2}\,\theta(x)$ and parameter m varying in the range $[0,1]$, and a second class of four {cn, nc, sd, ds} with argument $u(x) = (1-2m)^{-1/2}\,\theta(x)$ and parameter in the range $[0,1/2]$. The logarithmic derivative of these functions have simple poles on the relevant ínterval of the real axis at the origin or at $\pm K(m)$, where $K(m)$ is the complete elliptic integral of the first kind. A pole at $u=0$ would correspond to a singularity on the hyperplane $x.v = 0$. Eliminating this possibility we are left with $\{\mathrm{cn},\mathrm{nc}\,|\,m \in [0,1/2]\}$ and $\{\mathrm{cd},\mathrm{dc}\,|\,m \in [0,1]\}$ which have poles at $\pm K(m)$. In the first case $K(m)$ is always smaller than $(1-2m)^{-1/2}\pi/2$ so that a singularity is always present in region I. These solutions are thus also eliminated and we are left with the two functions $\mathrm{cd}(u,m)$ and $\mathrm{dc}(u,m)$. It was shown[3] that $K(m)$ is larger than $(1+m)^{-1/2}\pi$ for $m_c < m \leq 1$, where m_c ■ 0.827 is the solution of $K(m_c) = (1+m_c)^{-1/2}\pi$. This means that $\mathrm{cd}(u,m)$ and $\mathrm{dc}(u,m)$ have no singularity for these values of the parameter in the whole of Minkowski space, for timelike v, and that the only singularity, for spacelike v, is located on the two-dimensional singularity hyperboloid described above.

When the parameter m varies in the interval $[0,m_c]$ we have a singularity in region II, but we still can allow for such a solution if we restrict space-time to be conforma1 to the region I of Minkowski space. The physically acceptable solutions are thus

$$E_1(x) = \mathrm{cd}(a^{-1/2}\,\theta(x),m)$$

with

$$a = 1+m\,,\quad b = -2m,\quad c = 1$$

and its inverse

$$E_2(x) = \mathrm{dc}(a^{-1/2}\,\theta(x),m)$$

with

$$a = 1+m,\quad b = -2\,,\quad c = m.$$

A first type of solution is obtained taking the conformal factor equal to the meron function of section 3:

$$\rho(x) = f(x) \tag{12a}$$

The contortion scalar function corresponding to this choice is

$$\phi(x) = E_{1,2}(x) \tag{12b}$$

and the constant μ of eq (8a) takes the value

$$\mu_1 = 2m\ (1+m)^{-1}\ R/6$$

$$\mu_2 = 2\ (1+m)^{-1}\ R/6$$

The curvature scalar of such a space-time is given by

$$R = 24\ v^2/A^2$$

In order ttiat the relation (8c) be satisfied, the curvature scalar R, the coupl irig constant A and the parameter m have to obey the relation

$$\lambda R/6 = (1+m)^2/4m \tag{12c}$$

In the limit when m tends to 0, $\mathrm{dc}(\theta,m)=\sec\theta$, and in region I we have

$$\psi_2(x) = (1 - x^2/v^2)^{-1}\ A/|v^2|$$

The corresponding curvature $F_{ij}{}^{p}{}_{q}$ satisfies the duality condition of, ref.1 but the relation (12c) cannot be satisfied for finite λ.

The other type of solution is constructed taking $\psi = \rho\phi$ to be the rneron f'unction so that the conformal factor and the contortion scalar are given by:

$$\rho_{1,2}(x) = f(x)\ E_{1,2}(x) \tag{13a}$$

$$\phi_{1,2}(x) = (E_{1,2}(x))^{-1} = E_{2,1}(x) \tag{13b}$$

The corresponding curvature scalar is given by

$$R_1 = 12m\ (1+m)^{-1}\ 4v^2/A^2$$

$$R_2 = 12\ \ (1+m)^{-1}\ 4v^2/A^2$$

The relation between curvature scalar, coupling constant and parameter reads

$$A\ R/6 = 2m\ (1+m)^{-2} \tag{13c}$$

In the limit when m tends to 0 the solution $\rho_2(x)$ corresponds to a de Sitter space with curvature R = 48 v^2/A^2.

Choosing A = $|v|^2$ we recover a standard form of the de Sitter metric

$$\rho(x) = (1 - R/48\ \eta_{ij}\ x^i x^j)^{-1}$$

and the torsion scalar is $\phi_2(x) = \cos\theta(x)$.

In this limit eq (13c) is only satisfied for zero coupling λ.

5. DISCUSSION

To investigate the physical significance (if any) of the obtained solutions it is necessary to study the motion of a test particle in the resulting fields according to the equations of motion obtained in reference 1.

When the fourvector v is spacelike the fields have a singularity which has the same topology as the singularity of the Kerr metric. It is generally expected that the Kerr metric should play an important part in the description of the interaction of matter with intrinsic angular momentum. The ansatz on the conformal flatness of space-time naturally excludes this type of metric.

Our attempts to use a general Kerr-Schild ansatz forthemetric were unsuccessful. An investigation of the conformal transformation properties of the theory seems to indicate that the success of the 't Hooft--Corrigan-Fairlie-Wilczek ansatz is intimately linked withthe conformal flatness of the considered space-time.

A way out migth be found if there are non-trivial solutions of eq (6b).

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Resumo

Apresentamos uma discussão mais detalhada de uma certa classe de soluções exatas, no vazio, das equações que propusemos recentemente para a teoria de calibre do grupo de Poincaré. A construção das soluções se baseia sobre hipóteses e métodos bem conhecidos na Física dos Instantons.