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# x- Versus y-Scaling in Non-Relativistic Deep Inelastic Scattering

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**Abstrart** We show, in the context of non-relativistic potential scattering, that the appropriate scaling variable for the deep inelastic region is riot the usual Bjorken one $x_{Bj} = Q^2/2M\nu$ but instead, the variable $y = (2m\nu-\vec{q}^{\,2})/2q$.

The y-scaling is shown to be obtained in a natural way by using the WKB approximation. Numerical results are presented comparing the approach to scaling in terms of $x_{Bj}$ and y.

## 1. INTRODUCTION

The deep inelastic scattering of leptons by hadrons is one of the most powerful tools for testing the hadronic structure. Since 1968 a great deal of experimental results have met in evidence a composite picture of hadrons. The data are compatible with the view of hadrons as being composed of point-like (structureless) constituents. One of the most important evidences for this is the so-called Bjorken scaling, which may be interpreted as reflecting the fact that, in the limit of large momentum transfers (very short distances), the point-like constituents behave as quasi-free particles; as a consequence, the cross-sectionsfor deep inelastic scattering appear to be independent of any scale of mass.

However, there is another striking feature coming up from these experiments: the constituents have never been observed in final states, in which only ordinary hadrons are found emerging frorn the scattering. So, the data confront us with an apparently ambiguous situation, i.e., the constituents when tested at short distances behave as quasi - free particles although they seem to be permanently confined in the interior of hadrons at large distances.

More recently Quantum Chromodynamics (QCD)[1] was proposed as a candidate for describing strong interactions, whose predictions have been shown to be in good agreement with experiment. Among the results that can be understood in QCD we mention the precocious scaling, reflecting the smallness of the QCD fundamental length parameter, $\Lambda$, and scaling violations in the deep inelastic scattering of neutral and charged leptons by hadrons[2]. However it is still not perfectly established that QCD confinesthe constituents to the interior of hadrons, as it would be expected in accordance with the experimental results[3].

Deep inelastic scattering processes may be treated in a non-relativistic Quantum Mechanicai framework, sincethere is a complete analogy among the variables that describe the processes in the relativistic as well ai in the non-relativistic treatnient. Making use of this analogy we may consider deep ineiastic scattering in a non-relativistic framework as a "laboratory" for testing the hypothesis of pernianent confinement and see if the latter would destroy Bjorken scaling. The permanent confinement is simulated non-relativistically by the assumption tha: the constituents are coníined in a potential that increases with the distance.

It is clear that the mentioned non-relativistic framework isnot indicated to be applied to high energy processes since thesearestrickly relativistic. However we find its applicability in ncclear' plysics, more specifically, in inelastic scattering of electrons by nuclei[4].

Many authors have focused their attention on the problem of deea inelastic scattering treated in a non-relativistic framework. Amongthem we find the work of G.B. West[5], which sets up the formalism for the non -relativistic treatment and obtciins the analogue, in this framework, of the Bjorken scaling ($x$-scaling). He also iritroduces the so-called $y$-scaling by means of this formalism. The work of P.M. Fishbane and M.T. Grisaru[6] analyses some specific examples of confining potential, obtaining the analogue of Bjorken scaling for them. J. Bellandi Filho[7] treats the problem of deep inelastic electron scattering by a two spinless bound state (whose interactions is supposed to be of the harmonic oscillator type) and observes the scaling behaviour of thestructure function in the Bjorken limit. Another work, by G.C. Marques and C. F. Wey Jr.[8], studies the scaling laws of Many Body Systems in close analogy with the relativistic case, taking recoil effects into account. Finally, we should mention the recent paper by J.D. Bjorken and H. S. Orbach[9], which is closer to the spirit of our present work, approaching the $y$-scaling behaviour of the structure function in a semi-classical way.

We should briefly mention the different roles of $x$- and $y$-scaling in the non-relativistic framework. The former is reached when we consider the Bjorken limit of the structure function $\nu W(\nu,\vec{q}^{\,2})$, $\nu$ and $\vec{q}^{\,2}$ being, respectively, the energy and squared three momentum transferred by the virtual photon (using the Born approximation). In this limit, the $x$-scaling is obtained in the form of a delta function whose argument depends exclusively on $x = \vec{q}^{\,2}/2M\nu$ and in this case $x$ is equal to the inverse of the number of constituents[5,6,7]. The problem with this non-relativistic version of Bjorken scaling is that it gives no information about the initial momentum distribution of the constituents in the interior of the target, differing considerably from the relativistic version which explicitly associates the scaling variable $x = -q^2/2M\nu$ ($q$ is now the four momentum transferred) to the fraction of hadronic momentum carried by its constituents. This is precisely the difference which will favour $y$-scaling in the non-relativistic limit. What we observe then is that $qW$ turns out to depend only on $y = (2m\nu-q^2)/2q$ ($m$ is the constituent mass) and this scaling behaviour is obtained in terms of a Fourier transform in momentum space of the (initial) ground state wave-function. Being so, it is possible for us to know the initial momentum distribution of the constituents inside the target.

This paper is organized as follows. In section 2 we compare the roles of $x$-scaling in the relativistic context and $y$-scaling in the non-relativistic framework. In section 3 we show a procedure which enables us to obtain $y$-scaling in a natual way, by considering the WKB approximation for the wavefunction of the final state. We also compare this procedure with the result obtained for an exact case - the harmonic oscillator potential. In section 4 we show the numerical results for two specific examples of confining potentials, comparing the approach to scaling in $x$ - and $y$-variables. Finally, in section 5, some comments are presented in conclusion.

## 2. x- SCALING VERSUS y- SCALING

In 1969, R.P. Feynman, J.D. Bjorken and E.A. Paschos introduced the Parton Model[10] in order to describe the hadronic structure as manifested in high energy collisions. This is a model inspired on Field Theory in which, whenever it is possible, all complications inherent to

Quantum Field Theory are duely neglected. The quanta of the fundamental fields are simply called partons and the conditions assumed in this model are such that the interaction among partons is switched off when they are tested by leptons in deep inelastic scattering.

The hypotheses made in the Parton Model are equivalent to consider as relevant for the inelastic lepton-nucleon scattering only the handbag diagram, as shown in Fig. 1.

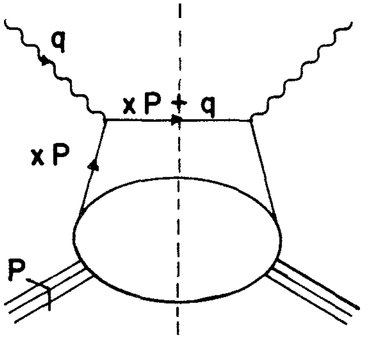


Fig. 1 - Handbag diagram for deep inelastic electron-nucleon scattering.

The cross-section obtained by means of this diagram is given by[11]:

$$\left(\frac{d^2\sigma}{dt\,du}\right)_{\ell N\to\ell N} = \frac{2\pi\alpha^2}{t^2}\left(\frac{s^2+u^2}{s^2}\right)\int dx \sum_i e_i^2\, x\, f(x)\, \delta\left[\left(\frac{t}{s+u}+x\right)(s+u)\right] \tag{2.1}$$

where $x$ is the fraction of the momentum carried by the struck parton; $e_i^2$ is the squared charge of the type $i$ parton ($\alpha$ is the fine structure constant); $f_i(x)$ is the probability to find a parton with momentum fraction between $x$ and $x+dx$, and finally, $s,t,u$ are the Mandelstam variables.

The argument of the delta function in (2.1) may be written as:

$$\delta\left[t + x(s+u)\right] = \delta(q^2 + 2M\nu . x) = \frac{1}{2M\nu}\,\delta\left(x - \frac{Q^2}{2M\nu}\right) \tag{2.2}$$

where we have considered the values assumed by the Mandelstam variables in the laboratory frame; $Q^2 = -q^2$ is the squared four momentum transferred by the virtual photon and $M$ is the nucleon mass. Remembering that in this frame $P.q = M\nu$ ($P$ is the nucleon four momentum), the argument of the delta function may be rewritten as:

$$\delta(q^2 + 2M\nu.x) = \delta\left[q^2 + 2M\nu.x + (xP)^2 - m^2\right] = \delta\left[(xP + q)^2 - m^2\right] \qquad (2.3)$$

where $m$ is the parton mass.

Equation (2.3) exhibits the condition that the struck parton remains on its mass-shell after the interaction. With the aid of equation (2.2) we may conclude that the fraction $x$ of the momentum $P$ carried by the parton, before it interacts with the virtual photon, is equal to the Bjorken variable $Q^2/2M\nu$. This shows explicitly that Bjorken scaling in the relativistic framework gives information about the initial momentum distribution of partons inside the hadron.

We now show that the analogue of Bjorken scaling in the non-relativistic framework is y-scaling. This is derived, as before, by supposing only the handbag diagram to be relevant for the scattering and by imposing that the partons remain on their "non-relativistic mass-shell" before and after being struck by the virtual photon.

In the non-relativistic framework, in the Born approximation, the structure function is written as[12]:

$$W(\nu,\vec{q}) = \sum_f |I_{f0}(q)|^2 \, \delta(E_f - E_0 - \nu) \equiv \sum_f |\langle\psi_f| \sum_i F_i(\vec{q}) e^{i\vec{q}.\vec{r}_i} |\psi_0\rangle|^2 \delta(E_f - E_0 - \nu) \qquad (2.4)$$

where $\psi_0$ is the (initial) ground state wave-function and $\psi_f$ is the (final) excited state wave-function to which the struck parton jumps after interacting with the virtual photon. The delta function imposes energy conservation ($E_0$, $E_f$ are the target initial and final energies); $F_i(\vec{q})$ is the constituent form factor which is equal to its charge in the structureless case,

When we write equation (2.4) in momentum space we observe that we need some knowledge about the potential. Supposing only that it does not depend explicitly on the constituents' velocity, it is possible to find a general form for (2.4). Due to the presence of the squared modulus in (2.4) we will find some interference terms which will not contribute in the limit of large $\vec{q}^2$ if the space ground state wavefunction is well behaved at the origin[12]. Being so, in the Bjorken limit, the structure function is reduced to:

$$\lim_{\vec{q}^2\to\infty} W(\nu,\vec{q}^2) = \lim_{\vec{q}^2\to\infty} \sum_i |F_i(\vec{q}^2)|^2 \int_{-\infty}^{+\infty} dk_{i_z} \; P_i(k_{i_z}) \; \delta\left[\nu - \frac{(\vec{k}_i+\vec{q})^2-\vec{k}_i^2}{2m_i}\right] \tag{2.5}$$

with

$$P_i(k_{i_z}) = \int \frac{d^2k_{i_\perp}}{(2\pi)^3} \; |f_i(k_{i_\perp},k_{i_z})|^2 \tag{2.6}$$

where $k_{i_z}$ and $k_{i_\perp}$ are the components of the constituents' three momenta along and perpendicular to the direction of the virtual photon, respectively; the factor $|f_i(k_{i_\perp},k_{i_z})|^2$ is related to the squared modulus of the ground state wave-function in momentum space, integrated over all momenta, except $\mathbf{k}_i$; $m_i$ is the $i$-th constituent mass.

Applying to the argument of the delta function the definition of the $y$-variable[5]:

$$y = \frac{2m\nu - \vec{q}^2}{2q} \tag{2.7}$$

we find

$$\delta\left[\nu - \frac{2k_{i_z} q + \vec{q}^2}{2m_i}\right] = \frac{m_i}{q}\,\delta\left[\frac{2m_i\nu - \vec{q}^2}{2q} - k_{i_z}\right] = \frac{m_i}{q}\,\delta(y-k_{i_z}) \tag{2.8}$$

Putting (2.8) into (2.5) and assuming that the constituents are point-like, we have:

$$\lim_{\vec{q}^2\to\infty} qW(\nu,\vec{q}^2) = \sum_i Q_i^2 \, m_i \, P_i(y) \; . \tag{2.9}$$

It should be noticed that the same equation (2.5) would be obtained had we considered only the non-relativistic analogue of the handbag diagram as relevant for the scattering, together with the condition that the parton should be on its "non-relativistic mass-shell"[13] before and after interacting with the virtual photon.

By means of equation (2.8) we notice that the y-scaling variable is the same as the component of the struck constituent initial momentum along the direction of interaction. And, finally, by equations (2.6) and (2.9) we see that the dependence of the structure function on the scaling variable y is related to the initial momenta distributionof the constituents inside the target, in close analogy with the relativistic version of Bjorken scaling.

## 3. WKB APPROXIMATION AND y- SCALING

We now show that the structure function $qW$, scaling in the variable $y$, may be obtained as a Fourier transform in momentum space of the ground state wave-function. As we shall see we get this result simply by isolating that part of the excited state wave-function which oscillates rapidly and by incorporating it into the factor $e^{iqx}$ - the non-relativistic electromagnetic current.

For simplicity we restrict our analysis to the one-dimensional case and look for the effects of the potential over just one of the confined particles.

Being so, we consider a spinless confined particle with mass $m$ and unit charge which interacts with the electromagnetic current, jumping from the (initial) ground state to an excited (final) state $n$. As we have seen before, the structure function is written in terms of the squared modulus of the transition amplitude $I_{f0}$, according to (2.4). In the special case of just one confined particle the transition amplitude is written as:

$$I_{f0} \equiv I_{n0} = \int_{-\infty}^{+\infty} dx\ \psi_0^*(x)\ e^{iqx}\ \psi_n(x)\ . \tag{3.1}$$

As the Bjorken limit involves high excitations, i.e., the final state level $n$ is very high, the excited wave-function may be approximated semi-classically[6]:

$$\psi_n(x) \simeq \frac{C}{\sqrt{p(x)}}\ \sin\left[\int_b^x p(x')dx' + \frac{\pi}{4}\right] \tag{3.2}$$

$$C = \sqrt{\frac{4m}{T(x)}}\ ;\qquad T(x) = \int_a^b \frac{2m}{p(x)}\ dx\ ;\qquad p(x) = \left[2m(E_n - V(x))\right]^{1/2} \tag{3.3}$$

where a and b are the classical turning points.

We assume that $\psi_0$ is effectively confined to a region $-c < x < c$, droppinp rapidly to zero outside this interval; c is a characteristic number of the potential (could be, in some cases, the classical turning points $x_0^{c\ell}$ corresponding to the ground state level) and this is supposed symmetric for simplicity.

We now suppose that the potential parameter is small such that the semi-classical approximation can be applied and consider $x_n^{c\ell} >> x_0^{c\ell}$

(this approximation breaks down for the square well case). With these hypotheses in mind we may expand the particle momentum, keeping only the first two terms:

$$p(x) = (\sqrt{2m\,E_n})\sqrt{1-\frac{V(x)}{E_n}} \simeq \sqrt{2m\,E_n} - \sqrt{\frac{m}{2E_n}}\,\tilde{V}(x) \tag{3.4}$$

where $\tilde{V}(x)$ is the maximum value of $V(x)$ in the interval $-c < x < c$.

Assuming that the excited level $n$ is very high and that $x_n^{c\ell} >> c$ then $\tilde{V}(x)/\sqrt{E_n} << 1$ in the interval $-c < x < c$. Therefore we may approximate $p(x)$ by the first term of the right-hand side of (3.4)[14]. With this approximation, the excited state wave-function given in (3.2) may be rewritten as:

$$\psi_n(x) \simeq \sqrt{\frac{4m}{T(x)}}\cdot\frac{1}{\sqrt{2m\,E_n}}\sin\left[\int_b^x \sqrt{2m\,E_n}\,dx' + \frac{\pi}{4}\right]$$

$$\simeq \sqrt{\frac{2}{T(x)E_n}}\sin\left[\sqrt{2m\,E_n}\;x + \phi\right]\;;\quad \phi = \sqrt{2m\,E_n}\,b + \frac{\pi}{4}\,. \tag{3.5}$$

By energy conservation we write $E_n - E_0 = \nu$ ; assuming that $E_n >> E_0$, then $E_n \simeq \nu$. Substituting this and (3.5) into (3.1), we obtain:

$$I_{n0} \simeq \sqrt{\frac{2}{T(x)E_n}}\int_{-\infty}^{+\infty} dx\;\psi_0^*(x)\,\frac{e^{iqx}}{2i}\left[e^{i\phi}\,e^{i\sqrt{2m\nu}} - e^{-i\phi}\,e^{-i\sqrt{2m\nu}}\right]. \tag{3.6}$$

When we take the Bjorken limit $(q,\nu\to\infty)$ the first term in brackets in (3.6), multiplied by $e^{iqx}$, oscillates rapidly giving a vanishing contribution to the integral, according to the Riemann-Lebesgue Theorem. But the second term, multiplied by $e^{iqx}$, may contribute **if** $q \sim \sqrt{2m\nu}$. This factor may be rewritten as:

$$q - \sqrt{2m\nu} = \frac{(q-\sqrt{2m\nu})(q+\sqrt{2m\nu})}{(q+\sqrt{2m\nu})} \simeq \frac{q^2 - 2m\nu}{2q} = -\,y \tag{3.7}$$

according to (2.7).

With the above considerations and the result (3.7), the modulus of the overlapping integral (3.6) may be written as:

$$|I_{n0}| \simeq \frac{1}{\sqrt{2T(x)E_n}}\left|\int_{-\infty}^{+\infty} dx\;\psi^*(x)\;e^{-iyx}\right| \tag{3.8}$$

As we have said at the beginning of this section, the $y$-scaling behaviour of the structure function $qW$ may be obtained by the procedure adopted above, i.e., by isolating the rapidly oscillating terms coming from the excited state wave-function and by incorporating them to the factor $e^{iqx}$. We see by (3.8) that the momentum variable is y. The $y$--scaling is reached in the form of a Fourier transform in momentum space of the ground state wave-function, making the linkage of the $y$ variable to the constituents' initial momentum distribution in the direction of the virtual photon.

In order to make clear the correctness of the above procedure, it is illustrative to analyse a particular potential for which the above result, (3.8), is obtained whithout the WKB approximation.

The confining potential in case is the three-dimensional harmonic $\frac{1}{2} m\omega^2 (x_1^2 + x_2^2 + x_3^2)$, where m is the mass of a spinless confined particle which is supposed to have unit charge; $\omega$ is the oscillator proper frequency. The n-th energy level is given by $E_n = \omega(n + \frac{3}{2})$ where $n = n_1 + n_2 + n_3$; the corresponding wave-function is given by:

$$\psi_{n_1 n_2 n_3}(\vec{r}) = \prod_{i=1}^{3} \psi_{n_i}(x_i) = \prod_{i=1}^{3} C_{n_i} \, e^{-\alpha^2 x_i^2/2} \, H_{n_i}(\alpha x_i) \tag{3.9}$$

where

$$\alpha = \sqrt{m\omega} \; ; \quad C_{n_i} = (m\omega/\pi)^{1/4} \left( 2^{n_i} \, n_i! \right)^{-1/2} \tag{3.10}$$

The structure function is obtained by means of the squared modulus of $I_{\hat{j}0}$ (equation (3.1)) provided we sum over all possible final states consistent with energy conservation. In terms of the wave-functions given in (3.9) the transition amplitude will be given by the product of thi-ee identical integrals of the form:

$$\begin{aligned} I_{n0} &= C_n C_0 \int_{-\infty}^{+\infty} dx \, e^{-\alpha^2 x^2} \, H_n(\alpha x) \, e^{iqx} \, H_0(\alpha x) = \\ &= (i)^n \left[ \frac{\vec{q}^{\,2n}}{2^n \alpha^{2n} n!} \right]^{1/2} e^{-\vec{q}^{\,2}/4\alpha^2} \end{aligned} \tag{3.11}$$

The structure function may then be written as:

$$W = \sum_{n_1 n_2 n_3} \left[ \frac{n!}{n_1! n_2! n_3!} \; q_1^{2n_1} \; q_2^{2n_2} \; q_3^{2n_3} \right] \frac{e^{-\vec{q}^{\,2}/2\alpha^2}}{n! 2^n \alpha^{2n}} \; \delta_{n + \frac{3}{2}, \frac{\nu}{\omega}} \tag{3.12}$$

Noticing that the above sum is a multinomial sum and including the density of states factor $dn/dE_n$, we get the following result for $|\vec{q}|W$ :

$$|\vec{q}|W = \frac{|\vec{q}|}{\omega n!}\left(\frac{\vec{q}^{\,2}}{2\alpha^2}\right)^n e^{-\vec{q}^{\,2}/2\alpha^2} \tag{3.13}$$

The right-hand side of the above equation is a Poisson distribution[7]. When we take the Bjorken limit, i.e., $\frac{4}{2\alpha^2} >> 1$ (and, cansequently, $n = \frac{\nu}{\omega} >> 1$) and use the Stirling approximation for $n!$, this distribution approachs a Gaussian one, with half-width $\sigma = \sqrt{\vec{q}^{\,2}/2\alpha^2}$:

$$qW \sim \frac{q}{\omega} \times \frac{1}{\sqrt{2\pi(\vec{q}^{\,2}/2\alpha^2)}} \cdot \exp\left[-\frac{1}{2} \cdot \frac{\left(\frac{\nu}{\omega} - \frac{\vec{q}^{\,2}}{2\alpha^2}\right)^2}{\vec{q}^{\,2}/2\alpha^2}\right]$$

Rearranging the argument of the exponential function into a suitable form and remembering the definition of the $y$-variable given in equation (2.7), we finally get:

$$qW \sim \frac{\alpha}{\omega\sqrt{\pi}} \exp\left[-y^2/\alpha^2\right] . \tag{3.14}$$

The result (3.14) is equal (except for a factor $1/m$) to the squared modulus of the Fourier transform of the ground state wave-function, in terms of one of the momentum components - for example, along the $z$-axis. This exhibits our earlier result (equation (3.8)) in a direct way

It should be mentioned that the same result was obtained for another particular potential by making use of a slightly different procedure. We have considered the linear one-dimensional confining potential ($V(x) = \infty$ for $x \leqslant 0$ and $V(x) = \alpha x$ for $x > 0$) and made a similar , but less dramatic hypothesis about the excited state wave-function. Once more rnaking use of the fact that Bjorken limit involves highexcitations, we have approximated the excited state wave function, intheoverllaping integral (3.1), by its asymptotic form. In this way y-scaling was obtained for $qW$.

**4. NUMERICAL RESULTS COMPARING x- AND y- SCALING**

In order to numerically study the approach to scaling in X- and y-variables we analyse in this section two specific examples of confining potentials: the harmonic oscillator and the square well. We begin with the former, since some relations referring to this case have been shown in the last section.

As we have seen, the structure function $qW(\nu,\vec{q}^2)$ is given by equation (3.13) and we must look at the way it approaches $y$-scaling. The non-relativistic version of Bjorken scaling is obtained by looking at the behaviour of the structure function $\nu W(\nu,\vec{q}^2)$ in the deep limit $\vec{q}^2\to\infty$, $\nu\to\infty$, with $x=\vec{q}^2/2m\nu$ finite. Its dependence on $\nu$, $\vec{q}^2$ is obtained by the same procedure adopted previously to obtain $qW$, and is written as:

$$\nu W = \frac{\nu}{\omega n!}\left(\frac{\vec{q}^2}{2\alpha^2}\right)^n e^{-\vec{q}^2/2\alpha^2} \tag{4.1}$$

with $n$ given by $n = \frac{\nu}{\omega} - \frac{3}{2}$ .

As commented in the Introduction, the structure function $\nu W(\nu,\vec{q}^2)$ approaches a delta function in the variable $x = \vec{q}^2/2m\nu$, as we take the Bjorken limit. With $\nu W$ in the form given above, for only one confined particle, we would observe the approximation to one branch of the delta function when considering progressively higher energies and momenta, since $x$ varies in the interval $0 \leqslant x \leqslant 1$. For this reason it would be better to consider the effect of the potential on two confined particles instead of one. In doing this, however, as we want to observe the approach to scaling, we should consider the interference terms appearing in the structure function (such terms come from the squared modulus of the transition amplitude in (2.4)). However, as discussed in section 2, as we take the Bjorken limit the contribution from these terms drops out rapidly to zero[12], remaining only the individual contributions. For this reason such interference terms will not be considered from the beginning.

In order to exhibit the form used on numerical calculations we write $y$, $\nu W(\nu,\vec{q}^2)$ and $qW(\nu,\vec{q}^2)$ in terms of the Bjorken variable $x =$ $= (\vec{q}^2/2M\nu)$, where $M = 2m$:

$$y = \frac{2m\nu - \vec{q}^2}{2m} = \frac{m\nu}{q}\,(1 - 2x) \tag{4.2}$$

$$\nu W = \frac{\vec{q}^2/2\alpha^2}{2x} \cdot \frac{e^{-\vec{q}^2/2\alpha^2}}{\left(\dfrac{\vec{q}^2/2\alpha^2}{2x} - \dfrac{3}{2}\right)!}\left(\frac{\vec{q}^2}{2\alpha^2}\right)^{\left(\frac{\vec{q}^2/2\alpha^2}{2x} - \frac{3}{2}\right)} \tag{4.3}$$

$$qW = K_1 \sqrt{\frac{\vec{q}^{\,2}}{2\alpha^2}} \; \frac{e^{-\vec{q}^{\,2}/2\alpha^2}}{\left(\frac{\vec{q}^{\,2}/2\alpha^2}{2x} - \frac{3}{2}\right)!} \; \left(\frac{\vec{q}^{\,2}}{2\alpha^2}\right)^{\left(\frac{\vec{q}^{\,2}/2\alpha^2}{2x} - \frac{3}{2}\right)} \qquad (4.4)$$

The factor $K_1 = (\sqrt{2m}/\alpha)$ has not been considered in numerical calculations.

The approach to scaling is investigated by attributing progressively higher values to $\vec{q}^{\,2}/2\alpha^2$, with $x$ varying in the interval $0 \leqslant x \leqslant 1$ and by observing the behaviour of $\nu W$ and $qW$. The results for $\nu W$ versus x are shown in Fig. 2 and for $qW$ versus $y/\alpha$ in Fig. 3

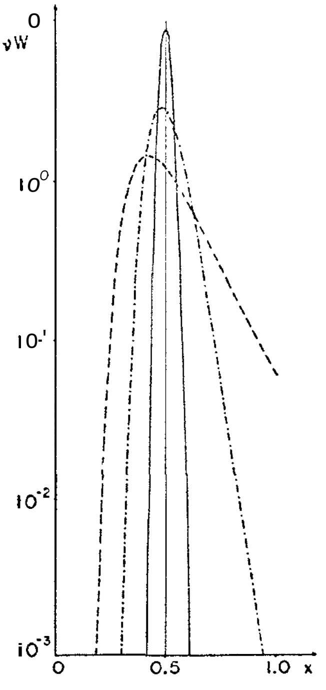


Fig. 2 - The plot shows the structure function $\nu W$ versus $x$, for the harmonic oscillator. The curves have been obtained for the following values of $\vec{q}^{\,2}/2\alpha^2$: --- 10; —.— 50; —— 500.

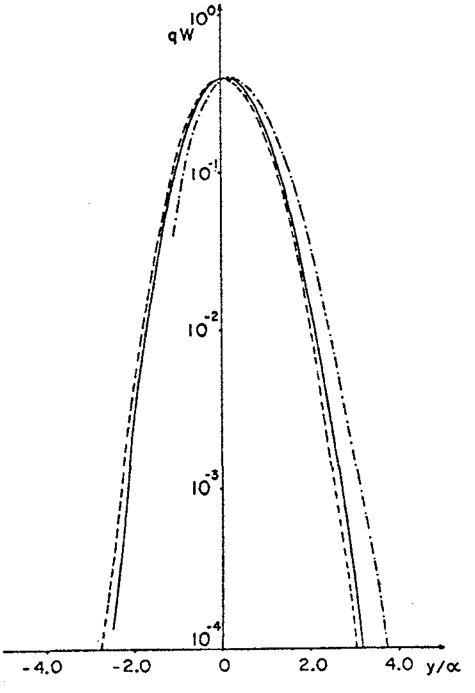


Fig. 3 - The plot shows the structure function $qW$ versus $y/\alpha$, for the harmonic oscillator. The curves have been obtained for the following values of $\vec{q}^{\,2}/2\alpha^2$: —.— 10; —— 50; --- 500.

We clearly observe from Fig. 2 that the structure function $\nu W$ approaches a sharp peak. By comparing figures 2 and 3 we may conclude that y-scaling is reached more rapidly than the non-relativistic analogue of Bjorken scaling, as would be expected.

Another numerical example may be considered. Now we supposethat the confining potential is simulated by a one-dimensional square well:

$$V(x) = \begin{cases} 0 & 0 < x < a \\ \infty & x \leqslant 0;\ x \geqslant a \end{cases} . \tag{4.5}$$

By solving the Schrödinger equation we obtain the following wave-function for just one spinless particle with unit charge and mass $m$:

$$\psi_n(x) = \sqrt{\frac{2}{a}}\ \sin\left(\frac{n\pi}{a}\,x\right) \tag{4.6a}$$

and

$$E_n = \frac{n^2\pi^2}{2ma^2} \tag{4.6b}$$

is the energy of the n-th level

Using the same procedure as before we find for the corresponding overlapping integral, the form:

$$I_{n1} = \frac{2i}{\pi}\left\{ e^{i(\frac{aq}{\pi}-n)\pi/2}\ \frac{\cos\left[(\frac{aq}{\pi}-n)\pi/2\right]}{1-(\frac{aq}{\pi}-n)^2} - e^{i(\frac{aq}{\pi}+n)\pi/2}\ \frac{\cos\left[(\frac{aq}{\pi}-n)\pi/2\right]}{1-(\frac{aq}{\pi}+n)^2}\right\} \tag{4.7}$$

As we are supposing this level to be very high ($n \gg 1$) and, for the square well potential, very high levels are very near, we may substitute the sum in the structure function by a density of states factor.

Now, considering as before that we may analyse the effect of the potential on two confined particles in the same way as in the one-particle case, we write the corresponding structure functions as:

$$\nu W = \frac{\alpha^2}{\pi^2 x\beta}\left\{\frac{\cos\left[\frac{\pi}{2}\ (\alpha-\beta)\right]}{1-(\alpha-\beta)^2} - \frac{\cos\left[\frac{\pi}{2}\ (\alpha+\beta)\right]}{1-(\alpha+\beta)^2}\right\}^2 \tag{4.8}$$

$$qW = K_2\ \frac{\alpha}{\pi\beta}\left\{\frac{\cos\left[\frac{\pi}{2}\ (\alpha-\beta)\right]}{1-(\alpha-\beta)^2} - \frac{\cos\left[\frac{\pi}{2}\ (\alpha+\beta)\right]}{1-(\alpha+\beta)^2}\right\}^2 \tag{4.9}$$

$$\alpha = \frac{aq}{\pi} \quad ; \quad \beta = \sqrt{\frac{\alpha^2}{2x} + 1} \qquad (4.10)$$

In the above relations we have applied energy conservation which relates $n$ to $\vec{q}^{\,2}$ and x by:

$$n = \sqrt{\frac{a^2\vec{q}^{\,2}}{\pi^2} \cdot \frac{1}{2x} + 1} \quad .$$

The constant $K_2 = \frac{4am}{\pi^2}$ was not considered in numerical calculations.

Proceeding as before, we have attributed progressively higher values to $\alpha = aq/\pi$, with $x$ in the interval $[0,1]$, obtaining the corresponding values for $\nu W$, $qW$ and $ay$. The results are shown in figures 4 and 5. In the latter, aiming at a better resolution, we have plotted the curves for $qW$ versus $ay$ only for ay < 10. For ay > 10 the curves present oscillations with progressively lower peaks compared to the central one (ar ay = 0).

By comparing figures 4 and 5 we observe that the same conclusions made in relation to figures 2 and 3 are valid: y-scaling is approached more rapidly than x-scaling.

**5. CONCLUSIONS**

In section 2 we have shown that y-scaling may be obtained by the non-relativistic analogue of the handbag diagram supposing that the struck parton remains on its energy-shell before and after interacting with the virtual photon. We have also shown that the y-scaling behaviour of the structure function explicitly displays some dependence on the initial momentum distribution of the constituents inside the target. These results show that y-scaling is a more suitable non-relativistic version of the well-known Bjorken scaling.

In section 3 we have shown a natural way of obtaining y-scaling using the information that the Bjorken limit involves high excitations, allowing us to approximate the excited state wave-functions by its correspondent WKB function. As a result we obtain the y-scaling form of the structure function as a Fourier transform in momentum space of the ground state wave-function, in terms of the variable $y$. This result stresses the fact that y-scaling gives us information on the initial

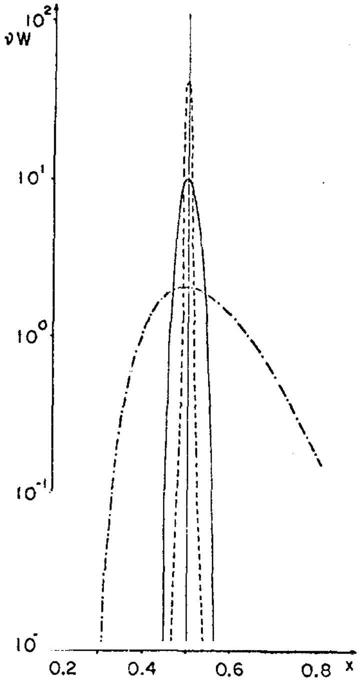


Fig. 4 - The plot shows the structure function $\nu W$ versus $x$ in the case of the infinite square well potential. The curves have been obtained for the following values of $aq/\pi$: —·— 10; —— 50; --- 200.

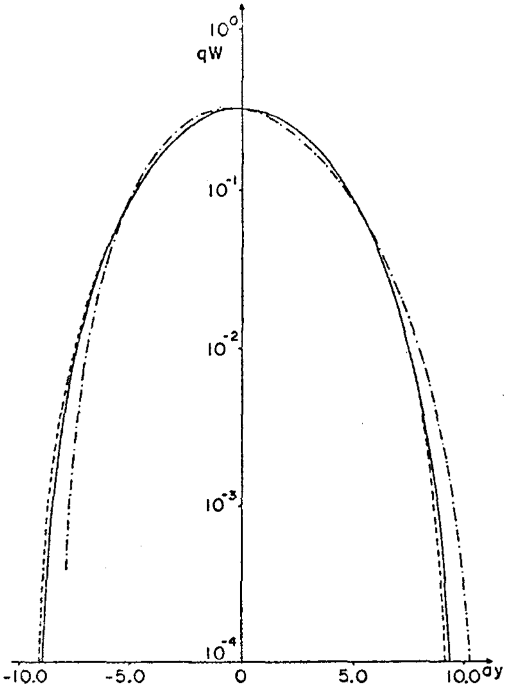


Fig. 5 - The same as Fig. 4 but for the structure function $\nu W$ versus ay.

momentum distribution of the constituents. We have also compared the result obtained in the general case with that for the harrnonic oscillator, for which no semi-classical approxirnation was needed.

Finally, in section 4 we have analysed the numericalapproach to scaling fcr two specific examples. Based on both of them we can seethat $y$-scaling is reached more rapidly than the non-relativistic version of Bjorken scaling.

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13. As a matter of fact, in the non-relativistic case we have an energy -shell, given by the condition: energy = (momentum)$^2$/2 mass.
14. This approxirnation is equivalent to supposing that, in the intervai $-c < x < c$, the excited level $\underline{n}$ is identical to the excited level of a symmetric square well, which is infinite for $|x| > c$.

### Resumo

No artigo mostramos que, no contexto do espalhamento não relativístico por potencial, a variável de "scaling" apropriada para a região profundamente inelástica não é a usual de Bjorken, $x_{B_j} = Q^2/2M\nu$, mas a variável $y = (2m\nu - \vec{q}^{\,2})/2q$.

Mostramos também que o "scaling" em $\underline{y}$ pode ser obtido de maneira natural, fazendo uso da aproximação WKB. Apresentamos ainda resultados numéricos que permitem comparar a aproximação ao "scaling" em termos de $x_{B_j}$ e y.