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The Exact Propagator of Time-Dependent Forced Harmonic Oscillator

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Abstract The work of Montroll[1] in deriving the propagator of time-dependent harmonic oscillator is generalized to obtain the propagator of time-dependent forced harmonic oscillator.

1. INTRODUCTION

The path integral was first introduced by Wiener[1] for the calculation of the mean values of certain functionals over the trajectories of a Brownian particle, and was later extended by Feynman[2] to the expression of the propagator, probability amplitude, in the configuration space of quantum mechanics[3,4,5]. The work of Cameron and Martin[6], Kac[7], and Montroll[8] for calculating some Wiener integrals can easily be applied to evaluate related Feynman integrals[9,10,11]. However, to our knowledge, the propagator of time-dependent forced harmonic oscillator has never been derived with these approaches. The purpose of this paper is to show that the work of Montroll can be generalized to obtain the exact propagator of time-dependent forced harmonic oscillator.

2. METHOD

In the path integral approach to nonrelativistic quantum mechanics the propagator, probability amplitude for a particle to go from the point $(\vec{x}_a, t_a)$ to the point $(\vec{x}_b, t_b)$, can be expressed as

$$K(\vec{x}_b, t_b; \vec{x}_a, t_a) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \exp\{(i/\hbar) \int_{t_a}^{t_b} L(\vec{x}, \dot{\vec{x}}, t)\, dt\}\, D\vec{x}(t) \qquad (2.1)$$

* Partially supported by CNPq.

where $L(\vec{x},\dot{\vec{x}},t)$ is the Lagrangian and $D\vec{x}(t)$ is designed to indicate that the integral is over all paths with fixed end points $(\vec{x}_a,t_a)$ and $(\vec{x}_b,t_b)$. We now assume that

$$L(\vec{x},\dot{\vec{x}},t) = \frac{m}{2}\dot{x}^2 - \frac{m}{2}\omega^2(t)x^2 + q(t)x \tag{2.2}$$

for one-dimensional time-dependent forced harmonic oscillator. By using Feynman's definition[2], the propagator of one-dimensional time-dependent forced harmonic oscillator is of the following form

$$K(x_b,t_b;x_a,t_a) =$$
$$= \lim_{n\to\infty} (m/2\pi i\hbar\tau)^{n/2} \int_{-\infty}^{\infty} .. \int_{-\infty}^{\infty} \exp\left\{(i\tau/2\hbar)\left[m\tau^{-2}\sum_{j=1}^{n}(x_j-x_{j-1})^2\right.\right.$$
$$\left.\left. - m\sum_{j=0}^{n-1}\omega_j^2x_j^2 + 2\sum_{j=0}^{n-1}q_jx_j\right]\right\} dx_1dx_2 \ldots, dx_{n-1} \quad . \tag{2.3}$$

For later convenience we have set $\tau = (t_b-t_a)/n$ and $r_j = r(t_a+j\tau)$ for any function $r(t)$. If we let $y_j = x_j(m/2\hbar\tau)^{1/2}$, then Eq.(2.3) can be rewritten as

$$K(x_b,t_b;x_a,t_a) = \lim_{n\to\infty} (i\pi)^{-n/2}(m/2\hbar\tau)^{1/2} \exp\left\{(i\tau/2\hbar)\left[m\tau^{-2}(x_a^2+x_b^2)\right.\right.$$
$$\left.\left. + 2q_0x_a - m\omega_0^2x_a^2\right]\right\}\int_{-\infty}^{\infty}\ldots\int_{-\infty}^{\infty} \exp\left\{i\left[\sum_{j=1}^{n-1}(2-\omega_j^2\tau^2)y_j^2\right.\right.$$
$$\left.\left. - 2\sum_{j=0}^{n-1}y_jy_{j+1} + (2\tau^3/m\hbar)^{1/2}\sum_{j=1}^{n-1}q_jy_j\right]\right\} dy_1dy_2\ldots dy_{n-1} \, . \tag{2.4}$$

Our basic Gaussian integral for the investigation of a quadratic Lagrangian is

$$\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} \exp\{i(y^TAy + 2b^Ty)\}\, dy \, ..dy_n =$$
$$= (i\pi)^{n/2}(\det A)^{-1/2}\exp(-ib^TA^{-1}b) \quad . \tag{2.5}$$

By comparing Eq. (2.4) and Eq. (2.5), we find that the matrix A is of the following form

$$A = \begin{vmatrix} a_1 & -1 & 0 & 0 & . & . & 0 & 0 & 0 & 0 \\ -1 & a_2 & -1 & 0 & . & . & 0 & 0 & 0 & 0 \\ 0 & -1 & a_3 & -1 & . & . & 0 & 0 & 0 & 0 \\ . & . & . & . & . & . & . & . & & \\ 0 & 0 & 0 & 0 & . & . & -1 & a_{n-3} & -1 & 0 \\ 0 & 0 & 0 & 0 & . & . & 0 & -1 & a_{n-2} & -1 \\ 0 & 0 & 0 & 0 & . & . & 0 & 0 & -1 & a_{n-1} \end{vmatrix} \tag{2.6}$$

with a[redacted] $2 - \omega_j^2 \tau^2$. The column matrix b has the following elements

$$b_1 = -y_0 + (\tau^3/2m\hbar)^{1/2} q_1 = - c\, \tau^{-1/2} x_a + a\, \tau^{3/2} q_1 ,$$

$$b_j = (\tau^3/2m\hbar)^{1/2} q_j = a\, \tau^{3/2} q_j \quad j = 2,3,\ldots,n-2 \tag{2.7}$$

and

$$b_{n-1} = -y_n + (\tau^3/2m\hbar)^{1/2} q_{n-1} = -c\, \tau^{-1/2} x_b + a\, \tau^{3/2} q_{n-1} ,$$

with $c = (m/2\hbar)^{1/2}$ and $a = (2m\hbar)^{-1/2}$. From the matrix A we define A_j and D_j

$$A_1 = a_1,\ A_2 = \begin{vmatrix} a_1 & -1 \\ -1 & a_2 \end{vmatrix} ,\quad A_3 = \begin{vmatrix} a_1 & -1 & 0 \\ -1 & a_2 & -1 \\ 0 & -1 & a_3 \end{vmatrix} ,\ \ldots,\ A_{n-1} = \det A ,$$

$$D_{n-1} = a_{n-1},\ D_{n-2} = \begin{vmatrix} a_{n-2} & -1 \\ -1 & a_{n-1} \end{vmatrix} ,\ D_{n-3} = \begin{vmatrix} a_{n-3} & -1 & 0 \\ -1 & a_{n-2} & -1 \\ 0 & -1 & a_{n-1} \end{vmatrix} ,\ \ldots,$$

$D_0 = \det A$.

They satisfy the following finite-difference equations[12]

$$A_j = a_j A_{j-1} - A_{j-2} \quad \text{and} \quad D_j = a_j D_{j+1} - D_{j+2} \tag{2.8}$$

with respectively boundary conditions $A_0 = 1$ and $D_n = 1$. Futhermore, it can easily be shown that

$$b^T A^{-1} b = \sum_{k=1}^{n-1} (D_k D_{k+1})^{-1} \Big(\sum_{j=k}^{n-1} b_j D_{j+1} \Big)^2 , \tag{2.9}$$

and

$$A_k = D_1 D_{k+2} \sum_{j=1}^{k+1} (D_j D_{j+1})^{-1} \tag{2.10}$$

when the matrix A is of form (2.6).

For $q(t) = 0$ (all $q_j = 0$), the path integral (2.4), which stands for the propagator of harmonic oscillator with time-dependent frequency, has been calculated by Montroll with the help of Eq. (2.5) through Eq. (2.10). In the next section we consider the case in which $q(t) \neq 0$.

3. EXACT PROPAGATOR OF TIME-DEPENDENT FORCED HARMONIC OSCILLATOR

By using Eq. (2.6) and Eqs. (2.7), Eq. (2.9) becomes

$$\begin{aligned} b^T A^{-1} b = {} & c^2 \tau^{-1} \{ (D_2/D_1) x_a^2 + (2/D_1) x_a x_b + x_b^2 \sum_{k=1}^{n-1} (D_k D_{k+1})^{-1} \} \\ & - (2ac\tau x_a/D_1) \sum_{j=1}^{n-1} q_j D_{j+1} - 2ac\tau x_b \sum_{k=1}^{n-1} (D_k D_{k+1})^{-1} (\sum_{j=k}^{n-1} q_j D_{j+1}) \\ & + \sum_{k=1}^{n-1} (D_k D_{k+1})^{-1} a^2 \tau (\sum_{j=k}^{n-1} q_j D_{j+1})^2 , \end{aligned} \tag{3.1}$$

after length but straightforward calculations. By substituting Eq. (3.1) into Eq. (2.5), we then obtain from Eq. (2.4) that

$$\begin{aligned} K(x_b, t_b; x_a, t_a) = {} & (m/2\pi i\hbar\tau \det A)^{1/2} \lim_{n\to\infty} \exp\{(im/2\hbar\tau)[(1 - D_2/D_1) x_a^2 \\ & - (2/D_1) x_a x_b + x_b^2 (1 - \sum_{k=1}^{n-1} (D_k D_{k+1})^{-1})]\} \exp\{i[(2ac\tau x_a/D_1) \sum_{j=1}^{n-1} q_j D_{j+1} \\ & + 2ac\tau x_b \sum_{k=1}^{n-1} (D_k D_{k+1})^{-1} (\sum_{j=k}^{n-1} q_j D_{j+1}) \\ & - a^2 \tau \sum_{k=1}^{n-1} (D_k D_{k+1})^{-1} (\sum_{j=k}^{n-1} q_j \tau D_{j+1})^2]\} . \end{aligned} \tag{3.2}$$

Here, the factor $\exp\{(i\tau/2\hbar)(2q_0 x_a - m\omega_0^2 x_a^2)\}$ in Eq. (2.4) has been assumed to be one in the limiting process as $\tau\to 0$ (or $n\to\infty$).

By converting Eq. (2.8) into differential equation, we find that

$$(D_{j+1} - 2D_j + D_{j-1})/\tau^2 = \omega_{j-1}^2 D_j$$

and in the limit as $\tau\to 0$

$$d^2D(t)/dt^2 = -\omega^2(t)D(t) \ , \tag{3.3}$$

where t takes the place of $t_a + j\tau$. The determinate $A(t)$ also would satisfy the same differential equation. However, from the boundary condition $D_n = D(t_b) = 1$ and $A_0 = A(t_a) = 1$, we see that A_j and D_j do not converge to functions $A(t)$ and $D(t)$, respectively. Now, by introducing two new functions $f(t)$ and $g(t)$ as

$$f_j = \tau D_j \quad \text{and} \quad g_j = \tau A_j \ ,$$

then we have in the limit as $\tau \to 0$

$$d^2f(t)/dt^2 + \omega^2(t)f(t) = 0 \qquad f(t_b) = 0 \quad \text{and} \quad \dot{f}(t_b) = -1 \tag{3.4}$$

$$d^2g(t)/dt^2 + \omega^2(t)g(t) = 0 \qquad g(t_a) = 0 \quad \text{and} \quad \dot{g}(t_a) = 1 \ , \tag{3.5}$$

Futhermore, the following identities can easily be shown[e] as $\tau \to 0$

$$(1 - D_2/D_1)/\tau = -\dot{f}(t_a)/f(t_a) \ , \tag{3.6}$$

$$1/\tau D_1 = 1/f(t_a) \tag{3.7}$$

and

$$\left[1 - \sum_{j=1}^{n-1} (D_jD_{j+1})^{-1}\right]/\tau = (1 - A_{n-2}/A_{n-1})/\tau = \dot{g}(t_b)/f(t_a) \tag{3.8}$$

since $\det A = A(t_b) = D(t_a) = f(t_a)/\tau = g(t_b)/\tau$. By substituting Eqs. (3.6), (3.7) and (3.8) into Eq. (3.2), we have

$$\begin{aligned} K(x_b,t_b;x_a,t_a) = {} & (m/2\pi i\hbar f(t_a))^{1/2} \exp\{(im/2\hbar f(t_a))\left[-x_a^2\dot{f}(t_a)\right. \\ & \left. - 2x_ax_b + x_b^2\dot{g}(t_b)\right]\} \lim_{n\to\infty} \exp\{i\left[(2ac\tau x_a/D_1)\sum_{j=1}^{n-1} q_jD_{j+1}\right. \\ & + 2ac\tau x_b \sum_{k=1}^{n-1} (D_kD_{k+1})^{-1} \sum_{j=k}^{n-1} q_jD_{j+1} \\ & \left. - a^2\tau \sum_{k=1}^{n-1} (D_kD_{k+1})^{-1} \left(\sum_{j=k}^{n-1} q_jD_{j+1}\right)^2\right]\} \ . \end{aligned} \tag{3.9}$$

With the help of Eqs. (2.10) and (3.8), we obtain as $\tau \to 0$

$$\tau\left(\sum_{j=1}^{n-1} q_jD_{j+1}\right)/D_1 = \left(\sum_{j=1}^{n-1} q_j\tau f_{j+1}\right)/\tau D_1 = (1/f(t_a)) \int_{t_a}^{t_b} q(\theta)f(\theta)d\theta, \tag{3.10}$$

$$\tau \sum_{k=1}^{n-1} (D_k D_{k+1})^{-1} \sum_{j=k}^{n-1} q_j D_{j+1} = (1/f(t_a)) \int_{t_a}^{t_b} q(\theta) g(\theta) d\theta \tag{3.11}$$

and

$$\tau \sum_{k=1}^{n-1} (D_k D_{k+1})^{-1} \left(\sum_{j=k}^{n-1} q_j D_{j+1}\right)^2 = (2/f(t_a)) \int_{t_a}^{t_b} \int_{t_a}^{\theta} q(\theta) f(\theta) q(\phi) g(\phi) d\theta d\phi \tag{3.12}$$

Eq. (3.11) and Eq. (3.12) have been shown in the Appendix A. Finally, by substituting Eqs. (3.10), (3.11) and (3.12) into Eq. (3.9), we have

$$\begin{aligned} K(x_b, t_b; x_a, t_a) = {} & (m/2\pi i\hbar f(t_a))^{1/2} \exp\{(im/2\hbar f(t_a)) [-x_a^2 \dot{f}(t_a) - 2x_a x_b + \\ & + x_b^2 \dot{g}(t_b)]\} \exp\{(i/\hbar f(t_a)) [x_a \int_{t_a}^{t_b} q(\theta) f(\theta) d\theta + \\ & + x_b \int_{t_a}^{t_b} q(\theta) g(\theta) d\theta \\ & - (1/m) \int_{t_a}^{t_b} \int_{t_a}^{\theta} q(\theta) f(\theta) q(\phi) g(\phi) d\theta d\phi]\} . \end{aligned} \tag{3.13}$$

For time-dependent forced harmonic oscillator with constant frequency ω, it can easily be shown (see Appendix B) that $g(t) = \omega^{-1} \sin \omega(t-t_a)$ and $f(t) = \omega^{-1} \sin \omega(t_b - t)$. Then Eq. (3.13) becomes

$$\begin{aligned} K(x_b, t_b; x_a, t_a) = {} & (m\omega/2\pi i\hbar \sin \omega T)^{1/2} \exp\{(im\omega/2\hbar \sin \omega T) [(x_a^2 + x_b^2) \cos \omega T \\ & - 2x_a x_b]\} \exp\{(i/\hbar \sin \omega T) [x_a \int_{t_a}^{t_b} q(\theta) \sin \omega(t_b - \theta) d\theta \\ & + x_b \int_{t_a}^{t_b} q(\theta) \sin \omega(\theta - t_a) \\ & - (1/m\omega) \int_{t_a}^{t_b} \int_{t_a}^{\theta} q(\theta) q(\phi) \sin\omega(t_b - \theta) \sin\omega(\phi - t_a) d\theta d\phi]\} \end{aligned} \tag{3.14}$$

with $T = t_b - t_a$.

Feynman and Hibbs obtain Eq. (3.14) by first showing that for quadratic Lagrangian, the propagator can be expressed as

$$K(x_b, t_b; x_a, t_a) = F(t_a, t_b) \exp\{i\, S_{cl}(x_b, t_b; x_a, t_a)/\hbar\} \tag{3.15}$$

and then by calculating the classical action $S_{cl}(x_b, t_b; x_a, t_a)$ and the normalization constant $F(t_a, t_b)$, respectively. However, we shall not consider the case $f(t_a) = 0$, which corresponds to the catastrophic phenomena (or focal points)[13,14], in this work.

4. CLASSICAL TRAJECTORY

From Eq. (2.2) the Lagrange's equation becomes

$$d^2x(t)/dt^2 + \omega^2(t)x(t) = q(t)/m, \qquad (4.1)$$

a nonhomogeneous second-order linear differential equation (without the first-order term). Before calculating the classical trajectory $\bar{x}(t)$, we would like first to study, the solution $f(t)$ of Eq. (3.4) and $g(t)$ of Eq. (3.5). By calculating the Wronskian of $f(t)$ and $g(t)$, we obtain[15]

$$W[f(t),\ g(t)] = g(t_b) = f(t_a) \neq 0$$

for all t. Therefore, they are two linearly independent solutions. Now, by assuming that[16,17]

$$g(t) = s(t)\ \sin[\gamma(t) - \gamma(t_a)]\ , \qquad (4.2)$$

where $s(t)$ and $\gamma(t)$ are the amplitude and the phase of a harmonic oscillator with time-dependent real frequency. In order to satisfy its boundary conditions, we must have

$$\ddot{s} - s^{-3}s^2(t_a) + \omega^2(t)s = 0 \qquad (4.3)$$

and

$$s^2(t)\dot{\gamma}(t) = s(t_a)\ . \qquad (4.4)$$

Futhermore, we obtain from $g(t)$

$$f(t) = s(t)\ \sin[\gamma(t_b) - \gamma(t)]\ , \qquad (4.5)$$

which has been shown in the Appendix B. By substituting Eqs. (4.2), (4.3), (4.4) and (4.5), we get

$$\begin{aligned} K(x_b,t_b;x_a,t_a) = {} & [m\dot{\gamma}(t_a)/2\pi i\hbar\ \text{rin}\ \Phi(t_b,t_a)] \\ & \exp\{[im\dot{\gamma}(t_a)/2\hbar\ \sin\Phi(t_b,t_a)]\ [(x_a^2 + x_b^2)\ \cos\ \Phi(t_b,t_a) \\ & - (\dot{s}(t_a)x_a^2 - \dot{s}(t_b)x_b^2)\ \sin\ \Phi(t_b,t_a) - 2\ x_a x_b]\} \\ & \exp\{[i\dot{\gamma}(t_a)/\hbar\ \sin\ \Phi(t_b,t_a)]\ [x_a \int_{t_a}^{t_b} s(\theta)q(\theta)\sin\Phi(t_b,\theta)d\theta \\ & + x_b \int_{t_a}^{t_b} s(\theta)q(\theta)\sin\ \Phi(\theta,t_a)d\theta \\ & -(1/m) \int_{t_a}^{t_b}\int_{t_a}^{\theta} s(\theta)s(\phi)q(\theta)q(\phi)\sin\Phi(t_b,\theta)\sin\Phi(\phi,t_a)d\theta d\phi]\} \end{aligned} \qquad (4.6)$$

Here, we have used the relations $s(t_a)\dot{\gamma}(t_a) = s(t_b)\dot{\gamma}(t_b)=1$, $s(t_a) = s(t_b)$ and $\gamma(t_a) = \gamma(t_b)$ and the notation $\Phi(x,y) = \gamma(x)-\gamma(y)$ for any two variaules x and y. For $q(t) = 0$, Eq. (4.6) is equivalent to Eq. (27) of Khandekar and Lawande[16]. For time-dependent forced harmonic oscillator with constant frequency ω, Eq. (4.6) can be reduced to Eq. (3.14) since $\gamma(t) = \omega t$ and $s(t) = \omega^{-1}$.

Since $f(t)$ and $g(t)$ are two linearly independent solutions of Eq. (4.1) with $q(t) = 0$, then the classical trajectory can be written as[18]

$$\bar{x}(t) = c_1 f(t) + c_2 g(t) + \int_{t_a}^{t} \frac{q(\theta)\left[f(\theta)g(t) - f(t)g(\theta)\right]}{W\left[f(\theta),\, g(\theta)\right]} d\theta \tag{4.7}$$

where c_1 and c_2 are two constants to be determined by the boundary conditions $\bar{x}(t_a) = x_a$ and $\bar{x}(t_b) = x_b$. With the help of Eqs. (3.4), (3.5), (4.2) and (4.5), we obtain

$$\begin{aligned}\bar{x}(t) = {} & \left[s(t)\dot{\gamma}(t_a)/\sin\Phi(t_b,t_a)\right] \{\left[x_a \sin\Phi(t_b,t) + x_b \sin\Phi(t,t_a)\right] \\ & + \left[\dot{\gamma}(t_a)/\gamma(t) \sin\Phi(t_b,t_a)\right] \left[\sin\Phi(t_b,t) \int_{t_a}^{t} s(\theta)q(\theta)\sin\Phi(\theta,t_a)d\theta \right.\\ & \left. - \sin\Phi(t,t_a) \int_{t_a}^{t} s(\theta)q(\theta)\sin\Phi(t_b,\theta)d\theta \right]\}\end{aligned} \tag{4.8}$$

after sirnplifications. Now, by calculating the classical action $S_{c1}(x_b,t_b;x_a,t_a)$ and the normalization $F(t_a,t_b)$, we show that Eq. (3.13) is of the form (3.15) as we expect.

APPENDIX A

By using Eqs. (2.10) and (3.8), we find that

$$\begin{aligned}\tau \sum_{k=1}^{n-1} (D_k D_{k+1})^{-1} \sum_{j=k}^{n-1} q_j D_{j+1} = {} & \tau \left\{ q_{n-1}D_n \sum_{k=1}^{n-1} (D_k D_{k+1})^{-1} \right.\\ & + q_{n-2}D_{n-1} \sum_{k=1}^{n-2} (D_k D_{k+1})^{-1} + \ldots + q_2 D_3 \sum_{k=1}^{2} (D_k D_{k+1})^{-1} \\ & \left. + q_1 D_2 (D_1 D_2)^{-1} \right\} \\ = {} & \tau\{q_{n-1}D_n(A_{n-2}/D_1 D_n) + q_{n-2}D_{n-1}(A_{n-3}/D_1 D_{n-1})\end{aligned}$$

$$+ \dots + q_2 D_3 (A_1/D_1D_3)$$

$$+ q_1 D_2 (A_0/D_1D_2)\} = (D_1\tau)^{-1} \sum_{j=1}^{n-1} q_j \tau\, g_{j-1}$$

$$\xrightarrow{\tau \to 0} (1/f(t_a)) \int_{t_a}^{t_b} q(\theta) g(\theta) d\theta \;. \qquad \text{(A.1)}$$

and

$$\tau \sum_{k=1}^{n-1} (D_k D_{k+1})^{-1} \left(\sum_{j=k}^{n-1} q_j \tau D_{j+1}\right)^2 = \tau^3 \{q_{n-1}^2 D_n^2 \sum_{k=1}^{n-1} (D_k D_{k+1})^{-1} + (2q_{n-1}D_n$$

$$+ q_{n-2}D_{n-1}) q_{n-2} D_{n-1} \sum_{k=1}^{n-2} (D_k D_{k+1})^{-1}$$

$$+ \dots + (2 \sum_{j=1}^{i-1} q_{n-j} D_{n-j+1} + q_{n-i} D_{n-i+1})\, q_{n-i} D_{n-i+1} \sum_{k=1}^{n-i} (D_k D_{k+1})^{-1}$$

$$+ \dots + (2 \sum_{j=1}^{n-2} q_{n-j} D_{n-j+1} + q_1 D_2 (D_1 D_2)^{-1} q_1 D_2 \}$$

$$= (\tau^3/D_1) \{q_{n-1}^2 D_n A_{n-2} + (2q_{n-1}D_n + q_{n-2}D_{n-1}) q_{n-2} A_{n-3}$$

$$+ \dots + (2 \sum_{j=2}^{n-1} q_{n-j} D_{n-j+1} + q_1 D_2) q_1 A_0 \}$$

$$= (\tau^3/D_1) \{ \sum_{j=1}^{n-1} q_j^2 D_{j+1} A_{j-1} + 2q_{n-1} D_n \sum_{j=2}^{n-1} q_{n-j} A_{n-j-1}$$

$$+ 2q_{n-2} D_{n-1} \sum_{j=3}^{n-1} q_{n-j} A_{n-j-1} + \dots + 2q_3 D_4 \sum_{j=n-2}^{n-1} q_{n-j} A_{n-j-1}$$

$$+ 2\, q_2 D_3 (q_1 A_0)\}$$

$$= (\tau D_1)^{-1} \{\tau \sum_{j=1}^{n-1} q_j^2 \tau f_{j+1}\, g_{j-1} + 2\tau q_{n-1} f_n \sum_{j=2}^{n-1} q_{n-j} \tau g_{n-j-1}$$

$$+ 2\tau q_{n-2} f_{n-1} \sum_{j=3}^{n-1} q_{n-j} \tau g_{n-j-1} + \dots + 2\tau q_3 f_4 \sum_{j=n-2}^{n-1} q_{n-j} \tau g_{n-j-1}$$

$$+ 2\tau q_2 f_3 (q_1 \tau g_0)\} \xrightarrow{\tau \to 0} (1/f(t_a)) \{\tau \int_{t_a}^{t_b} q^2(\theta) f(\theta) g(\theta) d\theta$$

$$+ 2 \int_{t_a}^{t_b} q(\theta) f(\theta) d\theta \int_{t_a}^{\theta} q(\phi) g(\phi) d\phi \}$$

$$= (2/f(t_a)) \int_{t_a}^{t_b} q(\theta)f(\theta)d\theta \int_{t_a}^{\theta} q(\phi)g(\phi)d\phi \;. \qquad \text{(A.2)}$$

APPENDIX B

Since $f(t)$ and $g(t)$ are two linearly independent solutions of Eq. (4.1) with $q(t) = 0$, we then have[15]

$$f(t) = W[\dot{f}(t_b),g(t_b)]\; g(t) \int_{t_b}^{t} g^{-2}(\theta)d\theta \;. \qquad \text{(B.1)}$$

By substituting Eq. (4.2) into Eq.(B.1), we obtain

$$f(t) = [g(t_b)s(t) \text{ rin } \Phi(t,t_a)] \int_{t}^{t_b} \{s(\theta) \sin \Phi(\theta,t_a)\}^{-2} d\theta$$

$$= [g(t_b)s(t) \sin \Phi(t,t_a)/s(t_b)] \int_{t}^{t_b} \{\dot{\gamma}(\theta)/\sin^2\Phi(\theta,t_a)\}d\theta$$

$$= [s(t)\sin \Phi(t_b,t_a)\sin\Phi(t,t_a)] \int_{\phi(t)}^{\phi(t_b)} [2/(1-\cos 2\phi)]d\phi \quad (\gamma(\theta)-\gamma(t_a)\equiv\phi(\theta))$$

$$= [s(t)\sin \Phi(t_b,t_a)\sin \Phi(t,t_a)]\,[\cot \Phi(t,t_a) - \cot \Phi(t_b,t_a)]$$

$$= s(t)\; [\sin \Phi(t_b,t_a)\cos \Phi(t,t_a) - \cos \Phi(t_b,t_a) \sin \Phi(t,t_a)]$$

$$= s(t) \sin \Phi(t_b,t) \;. \qquad \text{(B.2)}$$

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11. Although Wiener integral and Feynman integral are similar, however the latter cannot be built from bounded measure.

12. For derivation details, see Ref. (8), Section III.

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RESUMO

O trabalho de Montroll para deduzir o propagator do oscilador harmônico dependente de tempo é generalizado para obter o propagator do oscilador harmônico forçado também dependente de tempo.