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Torsion Generated Field in a Four-Dimensional Weitzenböck Space

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Abstract The field generated by torsion in a four-dimensional Weitzenböck space is considered. We interpret this field as a dynamical system with a quadratic Lagrangian density of the Yang-Mills form. The interaction with a spin 1/2 Dirac field is treated. The linear approximation for the free field generated by torsion is studied.

1. INTRODUCTION

The geometry of a four-dimensional manifold with a parallel field of local reference frames was considered in the literature as a possible model for an unitary field theory[1]. The geometric properties of this space, which is called as the Weintzenböck space[2], was later re-considered as the basic framework of a theory of gravitation[3].

In this space, the basic geometrical object is the torsion, and is the existence of this tensor field, which prevents the space of being globally flat, in the sense that exists a certain curvature tensor, build up from the knowledge of the torsion, at all points of the manifold. The geometry of this space is associated with an internal structure defined by the vierbeins, in such way that all internal Lorentz rotations (transformations of the group $SO(3,1)$) are carried out with constant matrices (global Lorentz transformations). This property, which resembles an anologue feature of the Minkowski space, has some interesting consequences since it brings out the presence of the six parameter Lorentz group. However, from the point of view of the interaction with fermions, it produces the result that no minimal coupling occurs in the sense of a gauge theory, since no connection may be defined. This means that the only correction to be made in the free Dirac equation for this case is the transition $\gamma_a \rightarrow \gamma_\mu(x)$ where γ_a are the four Dirac matrices. Presently, we propose a re-interpretation of this result in such form that is defined a covariant interna1 derivative in a weaker sense than used in gauge theories. The effect of the introduction of this derivative is to add two separate covariant terms. This procedure is consistent with the basic equations which determine the geometry of the manifold. As a result, the Dirac particle is "minimally coupled" to field of torsion in a form which is essentially similar to the method used in general relativity for the determination of the Fock-lvanenko coefficients.

In order to make the torsion a dynamical element, we propose a Lagrangian density of the standard Yang-Mills form and take the Hilbert variation on the vierbeins. The coupled system of this field, which we may call as the field generated by torsion, with a Dirac particle is considered. Since the resulting field equations are highly non-linear we consider the simple structure obtained from a weak-field approximation and discuss some properties of such approximation.

The theory which is obtained by this process may describe "gravitational effects" at microscopical level, where eventually the torsion is of more interest than the curvature which describes gravitation at macroscopic regions.

2. THE FIELD OF VIERBEINS

The vierbeins e^{a}_{μ} are the objects which transform the metric $g_{\mu\nu}$ into the local Lorentzian metric η_{ab} according to the well known relation

$$\eta_{ab} = e^{\mu}_{a} e^{\nu}_{b} g_{\mu\nu} \tag{2.1}$$

They satisfy the properties

$$e^{\mu}_{b} e^{a}_{\mu} = \delta^{a}_{b} \, , \; e^{a}_{\mu} \, e^{\nu}_{a} = \delta^{\nu}_{\mu}$$

In this paper the vierbeins e^{a}_{μ} and their reciprocal e^{μ}_{a} are the basic geometric quantities. They are restricted by the conditions

$$e^a_{\alpha;\nu} = \partial_\nu e^a_\alpha - \Gamma^\mu_{.\alpha\nu} e^a_\mu = 0 \tag{2.2}$$

Accordingly, the e^a_μ form a field of parallel fourvectors $e^0_\mu \ldots e^3_\mu$ in coordinate space. From *(2.2)* we ree that the connection $\Gamma^\mu_{.\nu d}$ asrociated to parallel transport is asymmetric. Another important consequence of *(2.2)* is that the curvature tensor associated to the connection $\Gamma^\mu_{.\alpha\nu}$ vanishes over all space

$$R^\mu_{.\nu\alpha\beta}(\Gamma) = 0 \tag{2.3}$$

Solving *(2.2)* for the $\Gamma^\mu_{\alpha\nu}$ one finds

$$\Gamma^\mu_{.\alpha\nu} = e^\mu_a \partial_\nu e^a_\alpha \tag{2.4}$$

Due to the asymmetric property of the $\Gamma^\mu_{.\alpha\nu}$, we can also consider the symmetric connection $\Gamma^\mu_{.(\alpha\nu)}$ as another possible connection, and introduce the derivative

$$e^a_{\mu;\nu} = \partial_\nu e^a_\mu - \Gamma^\lambda_{.(\mu\nu)} e^a_\lambda \tag{2.5}$$

Writing

$$\Gamma^\lambda_{.\mu\nu} = \Gamma^\lambda_{.(\mu\nu)} + S^\lambda_{.\mu\nu} \ , \tag{2.6}$$

$$S^\lambda_{.\mu\nu} = \Gamma^\lambda_{.[\mu\nu]}$$

We have from *(2.2)* and from the definition *(2.5)*

$$e^a_{\mu;\nu} = S^a_{\mu\nu} \tag{2.7}$$

where, for any internal object Ψ^a we have $\Psi^a = \Psi^\mu e^a_\mu$.

For the metric tensor $g_{\mu\nu}$ we have the metricity conditions

$$g_{\mu\nu;\alpha}(\Gamma) = 0 \tag{2.8}$$

These relations written for the symmetric part of the $\Gamma^{\mu}_{.\nu\alpha}$ takes the form

$$g_{\mu\nu;\alpha} = 2\, S_{(\mu\nu)\alpha} \tag{2.9}$$

A full covariant derivative of the e^{a}_{ν} may also be formally defined using the analogy with the internal structures associated to a Riemannian spacetime as:

$$e^{a}_{\mu|\alpha} = e^{a}_{\nu;\alpha} + \Omega^{a}_{.\,b\alpha} e^{b}_{\nu}$$

However, due to the conditions (2.2) and to the property that the e^{a}_{μ} form a set of linearly independent vectors in coordinate space, it follows that the internal connection Ω_{α} vanishes globally. This result means that the symmetry transformations in internal space are described by coordinate independent Lorentz matrices (global Lorentz transformations). Using the analogy with the Riemannian geornetry, or with the Riemann-Cartan geometry, where the existence of a curvature is associated with the presence of an internal curvature tensor, we look for a possible similar structure for the present situation. It is simple to show that the equations (2.3) and (2.6) conduct to the result that the curvature tensor associated to the symmetric part of the connection $\Gamma^{\mu}_{.\alpha\nu}$ is different from zero and is given in terms of the tensor of torsion $S^{\mu}_{.\alpha\nu}$ by (from now on we will indicate the symmetric part of the connection $\Gamma^{\mu}_{.\alpha\nu}$ by the symbol $C^{\mu}_{.\alpha\nu}$):

$$R^{\mu}_{.\nu\alpha\beta}(C) = S^{\mu}_{.\nu\beta;\alpha} - S^{\mu}_{.\nu\alpha;\beta} + S^{\lambda}_{.\nu\alpha} S^{\mu}_{.\lambda\beta} - S^{\lambda}_{.\nu\beta} S^{\mu}_{.\lambda\alpha} \tag{2.10}$$

Thus, the existence of a curvature $R^{\mu}_{,\nu\alpha\beta}(C)$ suggests that one must look for the corresponding internal structure. Defining the object

$$D^{a}_{.b\nu} = -S^{a}_{.b\nu} = \frac{1}{2}\left(\partial_{b} e^{a}_{\nu} - e^{\mu}_{b} \partial_{\nu} e^{a}_{\mu}\right) , \tag{2.11}$$

we introduce an "internal covariant derivative" by

$$\phi^{a}_{;\nu} = \partial_{\nu}\phi^{a} + D^{a}_{.b\nu}\phi^{b} \tag{2.12}$$

Due to the fact that the internal Lorentz transformations of the vierbein are carried out with constant matrices $L = (L^{a}_{.b})$, it follows that the two factors $\partial_{\nu}\phi$ and $D_{\nu}\phi$ in (2.12) transform separately as internal vectors. Thus, the above definition of the term $D_{\nu}\phi$ in (2.12) has the meaning of adding to the internal vector $\partial_{\nu}\phi$ the vector D_{ν}. Accordingly, D_{ν} is not formally interpret as a connection in internal space. For the e^{a}_{μ} which are the basic objects we define a "full covariant derivative" as

$$e^{a}_{\mu||\nu} = e^{a}_{\mu\,;\nu} + D^{a}_{.b\nu}e^{b}_{\mu} \tag{2.13}$$

we require that

$$e^{a}_{\mu||\nu} = 0 \tag{2.14}$$

The set of equations (2.13) and (2.14) is equivalent to the equations (2.7), since they imply in

$$\partial_{(\nu}e^{a}_{\mu)} - \Gamma^{\lambda}_{.(\mu\nu)}e^{a}_{\lambda} = 0$$

$$\partial_{[\nu}e^{a}_{\mu]} - S^{a}_{.\mu\nu} = 0 ,$$

which are again the definitions of the objects $\Gamma^{\lambda}_{.(\nu\mu)}$ and $S^{a}_{\nu\mu}$. From these considerations, we conclude that an "internal structure" may be associated to the symmetric part of the connection $\Gamma^{\lambda}_{.\mu\nu}$, in the sense that conditions (2.7) are preserved. These equations in terms of the metric assumes the form (2.9). In what follows we show that these conditions can be uniquely satisfied. Presently we deal with theequations (2.9) and (2.14). These two sets of equations are consistent if the Minkowski tensor satisfies the conditions

$$\eta_{ab;\nu} = - D^{c}_{.a\nu}\eta_{cb} - D^{c}_{.b\nu}\eta_{ac} = 2S_{(ab)\nu} \tag{2.15}$$

Accordingly, the traceof the object $D^{a}_{.b\nu}$ is different from zero, and has the value

$$D^{a}_{.a\nu} = - S^{a}_{.a\nu} = \frac{1}{2}\,(\partial_{a}e^{a}_{\nu} - e^{\mu}_{a}\partial_{\nu}e^{a}_{\mu}) \tag{2.16}$$

An "internal curvature" may be formally defined by the commutator

$$\phi^{a}{}_{;\nu;\sigma} - \phi^{a}{}_{;\sigma;\nu} = P^{a}{}_{.b\sigma\nu}(D)\,\phi^{b} \quad ,$$

where,

$$P_{\sigma\nu}(D) = \partial_{\sigma}D_{\nu} - \partial_{\nu}D_{\sigma} + [D_{\sigma},D_{\nu}] \tag{2.17}$$

In this formula we again call attention to the fact that $\partial_{[\sigma}D_{\nu]}$ and $[D_{\sigma},D_{\nu}]$ transform separately as internal tensors. From (2.10), (2.11), (2.14) and (2.17) one can show by a straightforward calculation that

$$P^{a}{}_{.b\rho\nu}(-S) = R^{a}{}_{.b\rho\nu}(C) + 2\,(S^{a}{}_{.c\rho}S^{c}{}_{.b\nu} - S^{a}{}_{.c\nu}S^{c}{}_{.b\rho}) \tag{2.18}$$

This formula differs from the corresponding expression for the Riemannian geometry by the presence of the extra term on the right hand side, quadratic in the vierbein components of the torsion. In matrix notation (2.18) takes the compact form

$$P_{\rho\nu}(-S) = R_{\rho\nu}(C) + 2\,[S_{\rho},S_{\nu}] \quad ,$$

where

$$R_{\rho\nu} = (R^{a}{}_{.b\rho\nu}) \text{ and } S_{\rho} = (S^{a}{}_{.b\rho}) \quad .$$

3. THE SPINOR REPRESENTATION

Following with the present treatment of an "internal structure" associated to the tensor $R^{\mu}{}_{.\alpha\beta\rho}(C)$, we consider the spinor representation of the object D_{ν} given by (2.11).

As usually one introduce the coordinate dependent Dirac matrices γ_{μ} by

$$\gamma_{\mu} = e^{a}{}_{\mu}\gamma_{a} \tag{3.1}$$

where γ_{a} are the constant Dirac matrices satisfying $\gamma_{(a}\gamma_{b)} = \eta_{ab}.1$.
Then,

$$\gamma_{\mu;\nu} = \gamma_{a}e^{a}{}_{\mu;\nu} = 0$$

which gives

$$\partial_\nu \gamma_\mu - \Gamma^{\lambda}_{.\mu\nu}\gamma_\lambda = 0$$

since

$$\gamma_{\mu;\nu} = \gamma_{\nu:\mu} - S^{\lambda}_{.\mu\nu}\gamma_\lambda = 0$$

we have

$$\gamma_{\mu:\nu} = S^{A}_{.\mu\nu}\,\gamma_\lambda \tag{3.2}$$

This equation is the spinor representation of (2.9). In sequence one introduces a "full covariant derivative" of the γ_μ by

$$\gamma_{\mu||\nu} = \gamma_{\mu:\nu} + [B_\nu,\gamma_\mu] \tag{3.3}$$

Using that $1.g_{\mu\nu} = \gamma_{(\mu}\gamma_{\nu)}$, and the equation (2.9) we have

$$\gamma_{\mu||\nu} = S_{\mu.\nu}^{\;\;A}\gamma_\lambda \tag{3.4}$$

Therefore, from (3.2), (3.3) and (3.4) one gets

$$[B_\nu,\gamma_\mu] = (S_{\mu.\nu}^{\;\;\lambda} - S^{\lambda}_{.\mu\nu})\gamma_\lambda \tag{3.5}$$

Solving (3.5) for the B_ν one finds

$$B_\nu = a_\nu.1 + \frac{1}{4i}\,S_{[\mu\lambda]\nu}\sigma^{\lambda\mu}_{..} \tag{3.6}$$

where

$$\sigma^{\lambda\mu} = e^{\lambda}_{a}e^{\mu}_{b}\,\sigma^{ab}_{..} = \frac{i}{2}\,e^{\lambda}_{a}e^{\mu}_{b}\,[\gamma^a,\gamma^b]\;.$$

The equation (3.6) can also be written as

$$B_\nu = a_\nu.1 - \frac{1}{4i}\,D_{[ab]\nu}\,\sigma^{ba}_{..} \tag{3.7}$$

This value for the "spinor connection" is formally similar to the expression of the Fock-Ivanenko coefficients, except that here the $D_{[ab]\nu}$ (the "connection" in the vierbein representation) is a linear function of the vierbein components of the torsion. The quantities in (3.7) remain undetermined. This fact allows for the preçence of charged spin $\frac{1}{2}$ wave functions. Indeed, defining the internal "covariant derivative" of a spinor[4] ψ by

$$\psi_{|\nu} = \partial_\nu\psi + B_\nu\psi \tag{3.8}$$

and forming the commutator

$$\psi_{|\nu|\sigma} - \psi_{|\sigma|\nu} = Q_{\sigma\nu}\psi$$

one finds for the "spinor curvature"

$$Q_{\sigma\nu} = \partial_\sigma B_\nu - \partial_\nu B_\sigma + [B_\sigma, B_\nu] \tag{3.9}$$

The contributions of the term proportional to the identity matrix in (3.7) to this curvature, has the form of an electromagnetic field strength. We recall that under internal transformations, $\psi' = S\psi$, the "spinor connection" B_ν transforms as

$$B'_\nu = SB_\nu S^{-1} ,$$

since presently S is a constant matrix. Using the expression (3.7) in this transformation one obtains

$$a'_\nu = a_\nu \tag{3.10}$$

$$b'_\nu = Sb_\nu S^{-1} \tag{3.11}$$

where b_ν is a rhort for $-\frac{1}{4i} D_{[ab]\nu}\sigma^{ba}_{\cdot\cdot}$. Due to the fact that presently the factors $\partial_\nu\psi$ and $B_\nu\psi$ transform separately as internal vectors (spinors), we see that it is possible to associate to the field B_ν a coupling constant. Similar argurnent holds for the a_ν, which is associated to the electric charge of the wave function \$. From (3.10) we conclude that $\partial_\nu\psi$ and $a_\nu\psi$ are disconected as long as the transformations with matrices S are considered. This property shows that a_ν is a connection associated to another internal transformation of ψ, namely the transformations of the unitary group $U(1)$. From the expression of the derivative given by (3.8) we see that the interaction term in the Lagrangian density for the wave function ψ is of the form

$$L_{int.} = \sqrt{-g}\, D_{[bc]a}\, \bar{\psi}\, \gamma^{a}_{\cdot}\sigma^{cb}_{\cdot\cdot}\, \psi \, . \qquad (3.12)$$

Using that

$$\gamma^{a}_{\cdot}\, \gamma^{c}_{\cdot}\, \gamma^{b}_{\cdot} = -\varepsilon^{acbd}_{\cdot\cdot\cdot\cdot}\, \gamma_d\, \gamma_5$$

if $a \neq c \neq b$, for sígnature (-2) and for $\gamma_5 = \gamma_0\gamma_1\gamma_2\gamma_3$, one gets

$$\gamma^{a}_{\cdot}\, \sigma^{cb}_{\cdot\cdot} = -i\, \varepsilon^{acbd}_{\cdot\cdot\cdot\cdot}\, \gamma_d\, \gamma_5$$

if $a = c$,

$$\gamma^{a}_{\cdot}\, \gamma^{c}_{\cdot}\, \gamma^{b}_{\cdot} = \eta^{ac}_{\cdot\cdot}\, \gamma^{b}_{\cdot} \, ,$$

clearly a, b, and c cannot be identical since $b \neq c$ in (3.12). Choosing a and c as the two possible identical values we have in thi's case

$$\gamma^{a}_{\cdot}\, \sigma^{cb}_{\cdot\cdot} = -i\, \varepsilon^{acbd}_{\cdot\cdot\cdot\cdot}\, \gamma_d\, \gamma_5 + \eta^{ac}_{\cdot\cdot}\, \gamma^{b}_{\cdot}$$

Substituting in (3.12)

$$L_{int.} = -i\sqrt{-g}\, \varepsilon^{acbd}_{\cdot\cdot\cdot\cdot}\, D_{[bc]}{}^{\bar{\psi}}{}_{a}\gamma_d\gamma_5\psi + $$

$$+ \sqrt{-g}\, D_{[bc]}{}^{c}_{\cdot}\, \bar{\psi}\, \gamma^{b}\psi \qquad (3.13)$$

With the vierbein components of the torsion we can from the vector of torsion

$$V_b = D_{[bc]}{}^{c}_{\cdot}$$

and the pseudo-vector

$$\sqrt{-g}\, W^{d}_{\cdot} = \varepsilon^{dbca}_{\cdot\cdot\cdot\cdot}\, D_{[bc]a}$$

The interaction term takes the form

$$L_{int} = igW^{d}_{\cdot} \ \bar{\psi} \gamma_d \gamma_5 \psi + \sqrt{-g} \ V_d \ \bar{\psi} \gamma^d \psi \qquad (3.14)$$

From (2.11) one finds

$$V_b = \frac{1}{4} \left[e^{\nu}_{b} \partial^{a}_{\cdot} - e^{\nu a}_{\cdot\cdot} \partial_b\right] e_{a\nu} \qquad (3.15)$$

$$W^{b}_{\cdot} = \frac{1}{\sqrt{-g}} \varepsilon^{bdca}_{\cdot\cdot\cdot\cdot} e^{\nu}_{a} \partial_c e_{d\nu} \qquad (3.16)$$

It is easy to verify that equations (2.17) and (3.15) imply that

$$V_b = \frac{1}{2} e^{\nu}_{b} \operatorname{Tr} D_{\nu}$$

Thus, the spinor current four-vector $\sqrt{-g} \ \bar{\psi} \gamma^d \psi$ is coupled to the four--vector formed with the trace of the "connection" D_ν. For a neutrino the Dirac equation takes the form

$$\gamma^a \partial_a \psi - \frac{1}{4i} (-i \sqrt{-g} \ W^{a}_{\cdot} \gamma_a \gamma_5 + V_a \gamma^a) \psi = 0$$

Thus, as usually, is also an eigenstate of γ_5 with eigenvalues $\varepsilon = \pm 1$.

4. THE FIELD EQUATIONS

Since the torsion is a bilinear function of the vierbeins and their first derivations, we shall postulate a Lagrangian density ofthe Yang-Mills form for the free field generated by the e^{a}_{μ}.

$$L_0 = \alpha \sqrt{-g} \ S^{\mu}_{\cdot \nu\alpha} \ S_{\mu}^{\cdot\cdot \nu\alpha}$$

$$= \alpha |e| \ S^{\mu}_{\cdot \nu\alpha} S_{\mu}^{\cdot\cdot \nu\alpha}$$

This choice is presently correct since the $S^{a}_{\cdot b\alpha}$ behave as a set of four internal tensors. The torsion has dimension L^{-1}, consequently a is a constant with dirnension $MT^{-2}L$. Using that

$$\delta |e| = |e| e^{\nu}_{a} \ \delta \ e^{a}_{\nu} \quad ,$$

and the expression of $S^{\mu}_{\cdot \nu\alpha}$ in terms of the vierbeins, one finds frorn

Hilbert's variational principle

$$\delta L_0 = \alpha |e| \, U^{\nu}_{a} \, \delta e^{a}_{\nu} = 0$$

where

$$U^{\lambda}_{a} \; \frac{1}{|e|} (|e| S_{a}^{\cdot\cdot\nu\lambda})_{,\nu} + \frac{1}{2} S^{\mu}_{\cdot\nu\alpha} S_{\mu}^{\cdot\cdot\nu\alpha} e^{\lambda}_{a} -$$

$$- 2 S^{\nu\lambda}_{\cdot\cdot\alpha} S_{\nu a}^{\cdot\alpha} = 0 \; . \qquad (4.1)$$

For the full Lagrangían density representing the interaction with a spin $\frac{1}{2}$ field we have

$$L = L_0 + L_D + L_{\text{int}} \; ,$$

where L_D is the free spin $\frac{1}{2}$ Lagrangian density. The field equations assume the form

$$U^{\lambda}_{a} - J^{\lambda}_{a} = 0 \; , \qquad (4.2)$$

and

$$e^{a}_{\mu} \gamma_{a} (\partial_{\mu}\psi - B_{\mu}\psi) + m\psi = 0 \qquad (4.3)$$

with,

$$J^{\lambda}_{a} = |e| \; \bar{\psi} \, N^{\lambda}_{a} \, \psi \; ,$$

where

$$N^{\lambda}_{a} = \frac{1}{2} \gamma^{A} \sigma^{sda}_{\cdot\cdot ea} \partial \lfloor d e_{S} \rfloor \alpha + \frac{1}{2} e^{\lambda}_{\cdot} \lceil_{s} \not\gamma e_{d} \rceil_{\alpha} \sigma^{sd}_{\cdot\cdot} e^{\alpha}_{a}$$

$$+ \frac{1}{2} \gamma^{c}_{\cdot} \sigma_{sa} (\partial_{c} e^{s\lambda}_{\cdot\cdot} - \partial^{s}_{\cdot} e^{\lambda}_{c})$$

where $\not\partial = \gamma^{a}\partial_{a}$. Using the previous decomposition of the product $\gamma^{b}_{\cdot}\sigma^{sd}_{\cdot\cdot}$ one finds'

$$N^{\lambda}_{a} = \frac{i}{2} \varepsilon^{qbsd}_{\cdot\cdot\cdot\cdot} (e^{\lambda}_{b} e^{\alpha}_{a} \partial \, [d e_{s}] \, \alpha + e^{\lambda}_{\cdot} [s \partial / b / e_{d}] \alpha \; e^{\alpha}_{\cdot a}$$

$$+ 2\partial_{[b}e^{\lambda}_{.s]}\eta_{ad})\gamma_q\gamma_5 + \frac{1}{2} e^{\alpha}_{.a} e^{\lambda}_{.} {}_{[\dot{s}\partial\dot{d}]} e_{s\alpha}\gamma_d$$

From Hilbert's variational principle

$$\bar{\delta} J_0 = \alpha \int \sqrt{-g}\, U^{\lambda}_{a}\, \bar{\delta}\, e^{a}_{\lambda}\, d_4x \ ,$$

considering variations $\bar{\delta}\, e^{a}_{\lambda}$ generated by infinitesimal coordinate transformations which vanish outside of a finite volume in space-time:

$$\bar{\delta} e^{a}_{.\lambda} = e^{a'}_{.\lambda}(x) - e^{a}_{\lambda}(x) = -\xi^{\alpha}_{,\lambda}\, e^{a}_{\alpha} - \xi^{\alpha}\partial_{\alpha} e^{a}_{\lambda}$$

We have

$$\bar{\delta} J_0 = - \alpha \int \sqrt{-g}\, U^{\lambda}_{a}\, (\xi^{\alpha}_{,\lambda} e^{a}_{\alpha} + \xi^{\alpha}\partial_{\alpha} e^{a}_{\lambda}) d_4x$$

$$= \alpha \int \left[(\sqrt{-g}\, U^{\lambda}_{a} e^{a}_{\alpha})_{,\lambda} - \sqrt{-g}\, U^{\lambda}_{a}\partial_{\alpha} e^{a}_{\lambda} \right] \xi^{\alpha}\, d_4x$$

Imposing invariance of the Action: $\bar{\delta}\, J_0 = 0$, one obtains

$$-(\sqrt{-g}\, U^{\lambda}_{a}\, e^{a}_{\alpha})_{,\lambda} + \sqrt{-g}\, U^{\lambda}_{a}\partial_{\alpha} e^{a}_{\lambda} = 0$$

Applying these conditions to the field equations (4.2), we have

$$(\sqrt{-g}\, J^{\lambda}_{a})_{,\lambda} - 2\sqrt{-g}\, J^{\lambda}_{b}\, S^{b}_{.\lambda a} = 0 \ . \qquad (4.4)$$

Since $\sqrt{-g}\, J^{\gamma}_{a}$ is a vector density of weight (+1), its divergence is equal to the covariant divergence with respect to the connection $\Gamma^{\mu}_{.(\nu\alpha)}$: $(\sqrt{-g}\, J^{\lambda}_{a})_{,\lambda} = (\sqrt{-g}\, J^{\lambda}_{a})_{:\lambda}$. Then, if we impose the subsidiary conditions

$$S^{b}_{.\lambda a}\, J^{\lambda}_{b} = 0 \qquad (4.5)$$

we get the conservation laws

$$(\sqrt{-g}\, J^{\lambda}_{a})_{:\lambda} = 0$$

Therefore, for the coupling of the field of torsion with spin $\frac{1}{2}$ systems, one has to postulate the subsidiary conditions *(4.5)* in order to have a conservation law for the current J^{λ}_{a}. The equations *(4.5)* are four conditions on the solutions of the field equations. From the form of $S^{b}_{\cdot\chi a}$ and of J^{λ}_{b}, we see that these conditions involve only non-linear terms in the vierbeins, the lowest order term being linear in the e^{a}_{μ} and quadratic in their first partial derivatives. Thus, in the first order approximation they are identically satisfied.

5. THE WEAK FIELD APPROXIMATION

In this section we consider the linearization of the field equations. As usually, this approximation corresponds to write $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, where the $h_{\mu\nu}$ are first order infinitesimals. In this case the vierbeins e^{a}_{μ} have to interpreted as *4x4* matrices. We use the notation $e = (e^{a}_{\cdot\mu})$. Then

$$g_{\mu\nu} = e^{T}{}_{\mu}{}^{a}_{\cdot}\,\eta_{ab}\,e^{b}_{\cdot\nu} = \eta_{\mu\nu} + 2\phi_{(\mu\nu)}$$

where we heve taken

$$e^{a}_{\cdot\mu} = \delta^{a}_{\mu} + \phi^{a}_{\cdot\mu} \quad . \tag{5.1}$$

Accordingly,

$$e_{a\mu} = \eta_{a\mu} + \phi_{a\mu}$$

and

$$e^{\mu}_{\cdot a} = \delta^{\mu}_{a} - \phi^{\mu}_{\cdot a}$$

The $(e^{\mu}_{\cdot a})$ is the inverse of the matrix $(e^{a}_{\cdot\mu})$. In this approximation there is no distinction between internal and spacetime indices. Presently the basic variables are the vierbeins with sixteen independent components, differently of what occurs in general relatively, where only ten of these components are of importance.

We have

$$S_{a\cdot\cdot}^{\ \nu\lambda} \equiv \frac{1}{2}\, g^{\nu\rho}\, g^{\lambda\sigma}(\partial_{\sigma}e_{a\rho} - \partial_{\rho}e_{a\sigma})$$

$$\approx \frac{1}{2}\eta^{\nu\rho}\eta^{\lambda\sigma}(\partial_\sigma\phi_{a\rho} - \partial_\rho\phi_{a\sigma})$$

Thus

$$\partial_\nu S_{a..}^{\nu\lambda} = \frac{1}{2}\eta^{\nu\rho}\eta^{\lambda\sigma}(\partial^2_{\sigma\nu}\phi_{a\rho} - \partial^2_{\rho\nu}\phi_{a\sigma})$$

$$= -\frac{1}{2}\eta^{\lambda\sigma}\Box\phi_{a\sigma} + \frac{1}{2}\partial^\lambda(\partial^\rho\phi_{a\rho}) \qquad (5.2)$$

In this approximation the quantities $S_{a\nu\lambda}$, or equivalently the $S^{a}_{\cdot\nu\lambda}$ are invariant under the gauge transformations of the vierbeins

$$\phi^a_{\cdot\mu} \to \phi^a_{\cdot\mu} + \frac{\partial\phi^a}{\partial x^\mu}$$

(the same argument holds true for the $S_a^{\cdot\cdot\nu\lambda}$). Therefore, we can impose four conditions on the e^a_μ [5]. Indeed, from Killing's equation, retaining only first order terms,

$$\delta\phi^a_{\cdot\mu} = \phi'^a_{\cdot\mu}(x) - \phi^a_{\cdot\mu}(x) = -\xi^a_{,\mu}$$

We choose these four conditions in such form that the divergence of the torsion contains only the propagation factor given by the D'Alembertian.

$$\partial^\rho\phi_{a\rho} = 0 \qquad (5.3)$$

Since presently the spacetime transformations are basically the Lorentz transformations, the conditions (5.3) are preserved for any frame of reference. Then

$$\partial_\nu S_{a..}^{\nu\lambda} = \frac{1}{2}\eta^{\lambda\sigma}\Box\phi_{a\sigma}$$

The gauge functions $\phi^a(x)$ are restricted to be solutions of the wave equation: $\Box\phi^a = 0$, similarly to the Lorentz gauge in special relativity.

The field equations (4.1) assume the form

$$\Box\, \phi_{a.}{}^{\lambda} = 0$$

in the absence of external sources. We also have from *(3.5)* and *(3.16)*

$$\begin{cases} V_b = \frac{1}{4}\left[\partial^a \phi_{ab} - \partial_b \phi\right] \\ W^b = \varepsilon^{bdca} \partial_c \phi_{da} \end{cases} \tag{5.4}$$

Note that one cannot use conditions *(5.3)* to simplify *(5.4)* since ϕ_{ab} is asymmetric. In *(5.4)* we used the notation $\phi = \phi^{a}_{.a}$. The sixteen components $\phi_{ab}(x)$ may be decomposed in the ten quantities $\phi_{(ab)}$ which contribute to lenght measurements, and in the six components $\phi_{[ab]}$ which contribute to the interaction with spin $\frac{1}{2}$ fields through the factor W^b. Among these components we have four conditions given by (5.3). We may write

$$\phi_{ab} = \chi_{ab} + \frac{1}{6}\eta_{ab}\phi + \phi_{[ab]} \tag{5.5}$$

where χ_{ab} is the symmetric trace free part of the ϕ_{ab}. Accordingly, from *(5.3)* we have

$$\partial^b \chi_{ab} + \frac{1}{4}\partial_a \phi + \partial^b \phi_{[ab]} = 0\ , \tag{5.6}$$

and

$$= \frac{1}{4}\left[\partial^a \chi_{ba} - \partial^a \phi_{[ba]} - \partial_b \phi\right] = -\frac{5}{16}\partial_b \phi - \frac{1}{2}\partial^a \phi_{[ba]} \tag{5.7}$$

$$W^b = \varepsilon^{bdca}\partial_c \phi_{[da]} \tag{5.8}$$

For the $\Gamma^{\lambda}{}_{(\mu\nu)}$ which may be written as $\Gamma^{a}_{.(bc)}$ we have

$$\Gamma^{a}_{.(bc)} = \frac{1}{2}\left(\partial_c \phi^{a}_{.b} + \partial_b \phi^{a}_{.c}\right) = \eta^{ad}\partial_{(c}\chi_{b)d} + \frac{1}{2}\delta^{a}_{(b}\partial_{c)}\phi +$$

$$\frac{1}{2}\eta^{ad}(\partial_c\phi_{[db]} + \partial_b\phi_{[dc]})$$

and the equation of the affine geodesic associated to the $\Gamma^a_{.[bc]}$ assumes the form

$$w_a + u^b\left(\frac{d}{du}\chi_{ab} + \frac{d}{du}\phi_{[ab]}\right) + \frac{1}{4}u_a\frac{d\phi}{du} = \Theta \tag{5.9}$$

u^a and w^a are defined with respect to an affine parameter λ as

$$w_a = \eta_{ab}\frac{d^2x^b}{d\lambda^2}$$
$$u^b = \frac{dx^b}{d\lambda}, \quad \frac{d}{du} = u^a\frac{\partial}{\partial x^a}$$

For the field equations one gets

$$\Box\,\chi_{ab} + \frac{1}{4}\eta_{ab}\,\Box\phi = 0 \tag{5.10}$$

$$\Box\,\phi_{[ab]} = \Theta \tag{5.11}$$

In the linear approximation the field generated by the vierbeins, which play the role of potentials in the variational principie is a mixing of spins 1 and 2 represented by the ten quantities χ_{ab}, ϕ, and by the six components $\phi_{[ab]}$. Among these quantities we have four constraints given by (5.6). The metric g_{ab} has the form

$$g_{ab} = \eta_{ab} + 2\phi_{(ab)}$$
$$= (1 + \frac{1}{2}\phi)\eta_{ab} + 2\chi_{ab}$$

The first derivatives of the gauge function $\xi^a(x)$ may be decomposed as

$$\xi_{a,b} = (\xi_{(a,b)} - \frac{1}{4}\eta_{ab}\xi_{c,}{}^{c}) + \frac{1}{4}\eta_{ab}\xi_{c,}{}^{c} + \xi_{[a,b]} =$$

$$= u_{(ab)} + \frac{1}{4}\eta_{ab}\xi_{c,}{}^{c} + \xi_{[a,b]}$$

Accordingly, the components χ_{ab}, ϕ and $\phi_{[ab]}$ are subjected to the gauge transformations

$$\bar{\delta}\,\chi_{ab} = -\mu_{(ab)}$$

$$\bar{\delta}\,\phi = -\,\xi_{a},{}^{a}$$

$$\bar{\delta}\,\phi_{[a,b]} = -\,\xi_{[a,b]}$$

The quantify $\phi = \phi_{a.}^{\ a}$ may be eliminated by taking $\phi(x) = 0$ in the original gauge frame along with the conditions $\xi_{a},{}^{a} = 0$. However, ξ_{ab} and $\phi_{[a,b]}$ cannot be eliminated (a similar result holds for the $\phi_{(ab)}$ in general relativity, since these components may be set equal to zero in the neighbourhood of a certain point x in a given coordinate system, but this result depends on the choice of such coordinates).

Thus, taking @=0 we obtain

$$\Box\,\chi_{ab} = 0 \tag{5.12}$$

$$\Box\,\phi_{[ab]} = 0 \tag{5.13}$$

$$\partial^{b}\chi_{ab} + \partial^{b}\,\phi_{[ab]} = 0 \tag{5.14}$$

$$V_{b} = \frac{1}{4}\,(\partial^{a}\chi_{ab} - \partial^{a}\,\phi_{[ba]})\ ,\ W^{b} = \varepsilon^{bdca}\partial_{c}\phi_{[da]} \tag{5.15}$$

The solutions of the equations (5.12) and (5.13) are of the form

$$\chi_{ab} = s_{ab}\, e^{ik_1 x}\ ;\ k_1^2 = 0\ ,\ s^{a}_{\ .a} = 0$$

$$\phi_{[ab]} = r_{[ab]}\, e^{ik_2 x};\quad k_2^2 = 0$$

From (5.14)

$$k_{1.}^{\ b}\, s_{ab} + k_{2.}^{\ a}\, k_{2.}^{\ b} r_{[ab]} = 0$$

Taking

$$k_1^a = (\frac{w_1}{c}, \frac{w_1}{c}, 0, 0), \quad k_2^a = (\frac{w_2}{c}, \frac{w_2}{c}, 0, 0)$$

it is easy to show that four amplitudes may be written as function of the remaining ones:

$$s_{11} = \frac{w_2}{w_1} r_{[01]} - s_{01} - s_{22} - s_{33}$$

$$s_{01} = \frac{w_2}{w_1} r_{[01]} - s_{11}$$

$$s_{02} = \frac{w_2}{w_1} (r_{[12]} + r_{[02]}) - s_{12}$$

$$s_{03} = \frac{w_2}{w_1} (r_{[03]} + r_{[13]}) - s_{13}$$

Thus, we have a mixing of spins 1 and *2* since the constraint equations (5.14) do not allow for a separation of these two quantities. This means that even in the weak field approximation the field generated by torsion cannot be separated into two distinct quantities. In other words, both χ_{ab} and $\phi_{[ab]}$ act simultaneously as potentials for the field $S^a_{\cdot\ bc}$. These potentials are coupled to spin $\frac{1}{2}$ particles through the expressions given by (2.15).

A comparison of the present results may be stablished withthe known expressions obtained for general relativity. Indeed, taking the expression for the Hamiltonian associated to the Dirac equation forthe present formulation and considering the non-relativistic transition, we can obtain a comparison with the energy spectrum which occurs for the same problem in general relativity. For this type of approximation the use of the Foldy-Wouthuysen representation is of interest[6]. Due to the formal similarity for the expression of the interaction terms in both theories, this comparison may be obtained in a simple form.

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2. R. Weitzenböck - *Invarianten Theorie* (Noordhoff, Groningen 1923).
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4. The Dirac spinor in the Weitzenböck space is defined in a form similar to the definition used in general relativity, namely, under arbitrary coordinate transformations we have $\psi'(x') = \psi(x)$ and under internal transformations generated by Lorentz rotations $\bar{\psi}(x)=S(L)\psi(x)$.
5. A similar result occurs in the linear approximation of Einstein's equation in general relativity, where these four conditions are usually written for the first order correction to the metric.
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RESUMO

O campo induzido pela torção em uma variedade, 4-dimensional, de Weitzenböck, é interpretada como um sistema dinâmico, dotado de uma Lagrangeana quadrática, do tipo Yang-Mills. A interação desse campo com um campo de spin 1/2, de Dirac, é tratada. A aproximação linear para o campo livre gerado pela torção, é construida.