Revista Brasileira de Física, Volume 13, n? 1, 1983

Galerkin Method and the Solution of the Orr-Sommerfeld Equation

R. F. S. ANDRADE

Instituto de Física, Universidade Federal da Bahia, Salvador, 40.000, BA, Brasil

Recebido em 9 de setembro de 1982

Abstract We use the method of Galerkin and a set of functions not previously reported to solve the Orr-Sommerfeld equation and to draw the stability diagram of the two-dimensional Poiseuille fiow with respect to two-dimensional disturbances in the (R,α) plane. The values obtained for R_{cr} and α_{cr} are in best accordance with those found in the literature.

1. INTRODUCTION

Variational methods are powerful tools for the solution of a great variety of physical problems which may be stated by means of an extremum principle. The best known of them is perhaps the method of Ritz. However one often faces oneself with problems which are not (at least apparently) derivable from an extremum principle and this prevents from the use of the above quoted variational methods. This is the case of severa1 problems of stability theory in hydrodynamics, from which the Orr-Sommerfeld equation is a very well known example[1].

The Galerkin method for solving differential equations is a very general one and indeed it may be shown to be equivalent to the Ritz method, when the considered equations are derivable from an extremum principle[2]. Appart of this possible identification, it belongs wlth others (tau, pseudospectral, colocation) to the set of the so-called spectral methods, which have been systematically investigated recently[3]. All of them have the common feature of searching the desired solution by expanding it into a complete set of functions. They differentiate themselves by the how the boundary conditions of the problem and the algebraic equations for the'expansion coefficients are treated. Among them, the method of Galerkin is certainly the simplest of all.

The aim of this article is to show how the solution of the Orr -Sommerfeld equation and the stability dlagram for the channel flowmay be given with the help of the Galerkin method. The solution of this problem, a classical one in the theory of hydrodynamic instabilities, has been searched since the beginning of the century. Consistent results have only been found in 1971, with the application of one of the above quoted methods[4]. Since then, several other works have confirmed these results[5-7]. Here we take the Galerkin method and a new expanding basis, not previously reported. The results we got agree and are highly consistent with those available in the quoted literature. Moreover we bring new features of the solution and of the employed method into light, and illustrate how a rather complex problem is simplified by the use of the Galerkin method.

In the following we have the Section 2, where the method of solutlon and the problem are formulated. The Section 3 brings the results we got by two different numerical procedures, and finally we discuss our results in comparision with others in the Section 4.

2. FORMULATION OF THE PROBLEM

We consider here the flow of an incompressible fluid between two infinite parallel fixed plates, which is caused by an external pressiire acting along the plates. Restricting ourselves to the study of the two dimensional problem, the velocity and pressure fieldsmay be writteri as $\vec{v}(x,z,t) = u(x,z,t)\hat{x}+w(x,z,t)\hat{z}$ and $p(x,z,t)$. The motion is described by the Navier-Stokes and continuity equations which read:

$$\begin{aligned}
&\frac{\partial u}{\partial t} + w\frac{\partial u}{\partial z} + u\frac{\partial u}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial z^2}\right)\\
&\frac{\partial w}{\partial t} + w\frac{\partial w}{\partial z} + u\frac{\partial w}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial z^2}\right) \qquad (1)\\
&\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0
\end{aligned}$$

ρ and ν in the equations (1) are the constant mass density and viscosity coefficient of the fluid. The null-velocity of the fluid at the plates localized at $z = \pm\frac{H}{2}$ implies in the boundary conditions for the solutions of (1):

$$u(x,z = \pm \frac{H}{2}, t) = W(x, z = \pm \frac{H}{2}, t) = 0 \tag{2}$$

The unique non-trivial time independent solution of (1) is

$$u = u(z) = U_m(1 - 4z^2/H^2)$$

$$w = 0 \tag{3}$$

$$\frac{\partial p}{\partial x} = - 8 \nu U_m/H^2$$

which characterizes the well known Poiseuille flow with parabolic velocity profil. However the system (1) may present other time dependent solutions which superimpose themselves to (3).

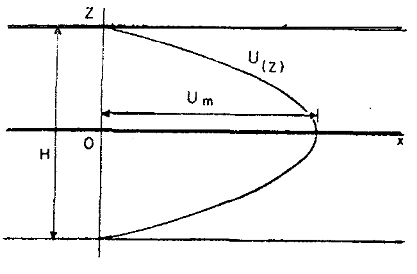

Fig.1 - The parabolic profil of the plane Poiseuille flow. H is the width ofthe channel and U_m the maximum velocity in the middle ofthe channel. The walls are localized at $z = \pm \frac{H}{2}$.

Before we look for these solutions, we introduce the stream function $\Psi(x,z,t)$, which is linked to the velocity field by

$$u = - \frac{\partial\Psi}{\partial z} \qquad w = \frac{\partial\Psi}{\partial x} \tag{4}$$

This is allowed, as the velocity field is two dimensional. It is also convenient to make a separation in the stream function into a time dependent and a time independent term, and write $\Psi(x,z,t) = \Psi_0(z) + \psi(x,z,t)$. $\Psi_0(z) = \frac{z^3}{3} - z$ reproduces the solution (3) when all lengthes and velocities are written in units of $H/2$ and U_m. The $\psi(x,z,t)$

is then governed by

$$\frac{\partial}{\partial t}\left[\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial z^2}\right]+\frac{\partial\psi}{\partial x}\left[2+\frac{\partial^3\psi}{\partial z^3}+\frac{\partial^3\psi}{\partial x^2\partial z}\right]-\left[z^2-1+\frac{\partial\psi}{\partial z}\right]\left[\frac{\partial^3\psi}{\partial x^3}+\frac{\partial^3\psi}{\partial x\partial z^2}\right]=$$

$$=\frac{1}{R}\left[\frac{\partial^4\psi}{\partial x^4}+2\frac{\partial^4\psi}{\partial x^2\partial z^2}+\frac{\partial^4\psi}{\partial z^4}\right] \tag{5}$$

The above equation was obtained after differentiating (1a) with respect to z, (1b) with respect to x and subtracting them. This causes the pressure terms to cancel each other in (5), where R is the adimensional Reynolds number R = $U_m H/2\nu$.

If we look for those solutions of (5) which are periodic in x, we may take care of the x-dependence not through a Fourier transform approach, but by the series

$$\psi(x,z,t) = \sum_{n=-\infty}^{\infty} \phi_n(z,t)\, e^{in\alpha x} \tag{6}$$

Although (6) does not represent the most general solution of (5), it has the advantage of reducing the x-dependence into infínite denumerable degrees of freedom, what constitutes a good start point for the further development of the problem. In (6) α is the (real) wave number and ϕ_n may in general be complex. The reality condition of the velocity field implies

$$\phi_n(z,t) = \phi^*_{-n}(z,t) \tag{7}$$

The equation system which governs the dependence of the Fourier coefficients upon the time t and the coordinate z are obtained if we insert (6) into (5). They read:

$$\frac{\partial}{\partial t}\left[\phi_n'' - n^2\alpha^2\phi_n\right] = \frac{1}{R}\left[\phi_n^{iv} - 2\,n^2\alpha^2\phi_n'' + n^4\alpha^4\phi_n\right] +$$

$$+\, in\alpha\left[-\,2\phi_n - (1 - z^2)(\phi_n'' - n^2\alpha^2\phi_n)\right] -$$

$$-\, i\alpha\sum_{m=-\infty}^{\infty}\left[(n-m)\phi_{n-m}(\phi_m''' - m^2\alpha^2\phi_m') - m\phi'_{n-m}(\phi_m'' - m^2\alpha^2\phi_m)\right] \tag{8}$$

Here the prime (') indicates the derlvatives of the ϕ_n with respect to z. The boundary conditions (2) are now to be written in terms of the Fourier components of the stream function and they become

$$\begin{aligned}&\phi_n(z = \pm 1,t) = \phi_n'(z = \pm 1,t) = 0 \quad &&\text{for } n \neq 0\\ &\phi_n(z = \pm 1,t) = 0 &&\text{for } n = 0\end{aligned} \tag{9}$$

The system (8) describes the most general flow realizations for the channel flow which are periodic in X. It consists also the starting point for the investigation of the basic Poiseuille flow, both by linear and non-linear methods[7]. Our purpose here is to make a linear stability analysis. The non-linear one still faces rather difficult conceptual questions and deserves further investigation , although some interesting features have already come into light.

The clue of the linear stability method is to assume that the velocity field superimposed upon the basic flow is somehow small. Due to this hypothesis we may neglect the terms which are non-linear in ϕ_n, so that the system (8) is transforrned into a linear one. This is certainly a very drastic simplification but the remaining equation is still a very complex problem to be solved. We look for eigensolutions of the linear equation by assuming for the ϕ_n a time dependence ofthe type $\phi_n(z,t) = e^{-\varepsilon t}\, f_n(z)$. This leads then to the known Orr-Sommerfeld equation:

$$\frac{1}{R}\left[f_n^{IV} - 2n^2\alpha^2 f_n'' + n^4\alpha^4 f_n\right] + in\alpha\left[-2f_n - (1-z^2)(f_n'' - n^2\alpha^2 f_n)\right] =$$

$$= \varepsilon\left[n^2\alpha^2 f_n - f_n''\right] \tag{10}$$

Since the different ϕ_n are not coupled any more in the linear version of (8), we will take hereafter n=l and drop the index.

The general solution for ϕ in terms of the eigenfunctions of (10) is then:

$$\phi(z,t) = \sum_i C_i\, f_i(z)\, e^{-\varepsilon_i t} \tag{11}$$

where C_i are constants dependingupon the Inicial conditions. It is clear that if Real$(\varepsilon_i) > 0$ for all ε_i, then every possible perturbation of the form (11) will decay with time. If only one (or more) ε_i does not satisfy this condition, then the perturbations will grow with time, and the basic flow is not stable anymore. The knowledge of the spectrum of (10) allows us to answer the question of the linear stability of the Poiseuille flow.

The equation (10) will be solved by employing the method of Galerkin. We expand the unknown function $f(z)$ in terms ofthefunctions of a given complete set, as in all spectral methods.

$$f(z) = \sum_k g_k\, G_k(z) \tag{12}$$

In the case of Galerkin's method, each function G_k must obey the right boundary conditions of the problem, what ensures that the sum (12) will also do. The problem is then reduced to finding the values of the coefficients g_k.

The expanding functions we use are the eigensolutions of $\frac{d^4h(z)}{dz^4} = a\, h(z)$ which satisfy the boundary conditions $h(z = \pm 1) = h'(z = \pm 1) = 0$. We can find the following solutions for the above equation:

$$S_n(z) = \frac{\cos d_n \cosh d_n z - \cosh d_n \cos d_n z}{\left[\cos^2 d_n + \cosh^2 d_n\right]^{1/2}}, \text{ with } \mathrm{tg}\, d_n + \mathrm{tgh}\, d_n = 0 \tag{13a}$$

$$A_n(z) = \frac{\cos e_n \sinh e_n z - \cosh e_n \sin e_n z}{\left[\cosh^2 e_n - \cos^2 e_n\right]^{1/2}}, \text{ with } \mathrm{tg}\, e_n - \mathrm{tgh}\, e_n = 0 \tag{13b}$$

The S_n and the A_n are orthogonal to each other in the interval $[-1,1]$ to normalized to unit. Similar functions have already been pointed out before'. They may be used to find respectively even and odd solutions of (10). We need to take only one subset at a time, since (10) does not couple functions of different parities.

We will take the set (13a), for it is known that (10) presents at most one eigenvalue with Real $\varepsilon < 0$, to which is associated an even eigenfunction. So we write

$$f(z) = \sum_n s_n S_n(z) \tag{14}$$

The next steps in the Galerkin method consist of inserting (13a) into (10), multiplying the whole equation by $S_m(z)$ and integrating in the interval $[-1,1]$, what leads to

$$\sum_n \left[(\alpha^2+d_n^4)\delta_{m,n} - 2 R_{m,n}\alpha^2 + i\alpha R\left[P_{m,n} - R_{m,n} - \alpha^2 Q_{m,n} + (\alpha^2-2)\delta_{m,n}\right]\right] s_n =$$

$$= \varepsilon \sum_n \left[\alpha^2\delta_{m,n} - R_{m,n}\right] s_n \tag{15}$$

or, in a short hand notation

$$\sum_n a_{mn}\, s_n = \varepsilon \sum_n b_{mn}\, s_n$$

In (15) the elements $P_{m,n}$, $Q_{m,n}$ and $R_{m,n}$ are defined by:

$$P_{m,n} = \int_{-1}^{1} S_m(z) z^2 \; S''_n(z) \; dz$$

$$Q_{m,n} = \int_{-1}^{1} S_m(t) \; z^2 \; S_n(z) \; dz \tag{16}$$

$$R_{m,n} = \int_{-1}^{1} S_n(t) \; S''_n(z) \; dz$$

and the eigenvalue ε has been scaled $\varepsilon R \to \varepsilon$. The above integrals may be easily performed, so that the elements in (16) are given in closed analytical expressions of d_n.

The linear system (15) may also be written in a matrix formas

$$A\vec{s} = \varepsilon \; B\vec{s} \tag{17}$$

where $(A)_{mn} = a_{mn}$ $(B)_{mn} = b_{mn}$ and $(\vec{s})_n = s_n$. To consider (17) as a diagonalization problemwe just have to multiply theequation by B^{-1} to get

$$B^{-1}A\vec{s} = \varepsilon\vec{s} \tag{17a}$$

The eigenvalues of (17a) agree with those of (10) and the components of the corresponding eigenvectors are the coefficients of the expansion of the Orr-Sommerfeld eigenfunctions in terms of the $S_n(z)$. Thus the application of the Galerkin method to this problem has led us to look for the solution of (17a), what is a completely equivalent problem to solving (10). At this polnt we would like to stress the conceptual simplicity of the employed method, what does not prevent its application even to a rather complex problem as the Orr- Sommerfeld equation.

3. RESULTS

Here we will discuss the spectrum of (10) which will be determined by the solution of (15a). The first point to be considered is that the equivalence between (10) and (15) is only reached when in the sums in (15) n runs from 0 to ∞. That means infinite matrices A and B in (17). As we are compeled to employ numerical methods to solve (17), the matrices A and B will be cut off. The approximate solutions we get will be discussed, as well as their dependence upon the number N of considered functions in the expansion (or the size of the cut off matrices A and B).

As a second point we recall that (10) depends upon two parameters: the wave number a and the Reynolds number R. As we have already mentioned, there exists at most one eigenvalue which may become critical (Real $\varepsilon < 0$) if we vary the pararneters a and R. The regions of stability and instability of the Poiseuille flow in the (R,α) plane are then limited by the curve

$$\text{Real}\,(\varepsilon_{crit}) = 0 \tag{18}$$

In particular we want to find the values of R_{cr} and a_{cr}, which are the coordinates of the point with the smallest value of R belonging to the curve (18).

Now we solve the equation (17a). First of all we must find B^{-1}. This can be done either numerically or, due to the special choice of our function basis, by giving the exact analytical form of its elements.

In the first case B^{-1} is determined by a numerical routine, and after multiplying it by A the resulting $B^{-1}A$ is diagonalized. The Table 1 presents some results for the value of R_{cr} and its dependence upon the number of considered functions in the Galerkin procedure. We see that the critical value of R_{cr} approaches a constant value (independent of N) as N increases. An interesting feature in Table 1 is that the constant value of R_{cr} is reached from only one side, as if we were

Table 1 - Dependence of the critical value of the Reynolds number upon the number N of considered functions in the Galerkin expansion. For each given N the real part of the critical eigenvalue changes sign in the interval $[R_{cr}, R_{cr} + 1]$. The real part of the critical eigenvalue is also presented.

N	R_{cr}	Critical Eigenvalue
20	5862	$.6984 \times 10^{-3}$
30	5783	$.3293 \times 10^{-2}$
40	5774	$.5754 \times 10^{-2}$
50	5772	$.8940 \times 10^{-2}$
60	5772	$.4743 \times 10^{-2}$
80	5772	$.2902 \times 10^{-2}$
90	5772	$.2691 \times 10^{-2}$

looking for the minimum of a variational problem. In Figure 2 we draw the curve of marginal stability as obtained in this firstcaseand

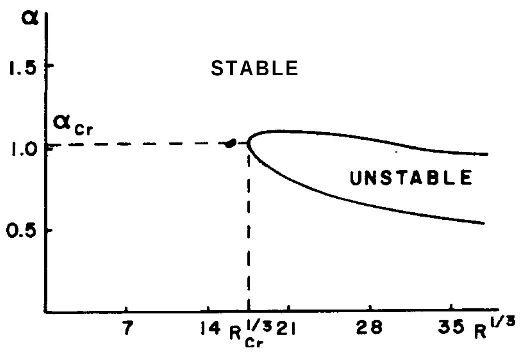

Fig.2 - The regions of stability and instability in the (R,α) plane for the plane Poiseuille flow, as given by the linear stability theory. The critical values of the parameters are $R_{cr} = 5772$ and $a_{cr} = 1.02$.

indicate the point of coordinates (R_{cr}, α_{cr}). The values $\alpha_{cr} = 1.02$ and $R_{cr} = 5772$ we have found are the "exact" numerical results as given in the literature.

If B^{-1} is to be determined analytically we first find a matrix U, such that $\tilde{B} = UBU^{-1}$ is diagonal. It may be eaçily shown[7] that if we define $g_m(z) = \cos(2m-1)\frac{\pi}{2} z$, then the matrix elements of U are given by

$$U_{mk} = \int_{-1}^{1} g_m(z) \quad S_k(z) \; dz \tag{19}$$

Since B is a symmetric matrix (in (16) $R_{mn} = R_{nm}$) it follows that $(U^{-1})_{mk} = U_{km}$. Thus the matrix elements of $\tilde{B}$, $\tilde{B}^{-1}$ and B^{-1} are

$$\tilde{B}_{mn} = \left[\frac{\pi^2}{4} (2m-1)^2 + \alpha^2\right] \delta_{m,k}$$

$$\tilde{B}^{-1}_{mk} = \frac{\delta_{m,k}}{\frac{\pi^2}{4} (2m-1)^2 + \alpha^2} \tag{20}$$

$$B^{-1}_{mk} = \sum_{n=1}^{\infty} \frac{U_{nm} \; U_{nk}}{\frac{\pi^2}{4} (2n-1)^2 + \alpha^2}$$

Multiplying B^{-1} (as given by the above expression) by equation (15) we get:

$$\sum_n \left[2\alpha^2\delta_{mn} + (d_n^4-\alpha^4)B^{-1}_{m,n} - i\alpha R(\delta_{m,n} - 2B^{-1}_{m,n} + \hat{P}_{m,n} - \alpha^2\hat{Q}_{m,n})\right] s_n = \varepsilon\, s_m \tag{21}$$

with the new matrix elements

$$\hat{P}_{mn} = \sum_{k=1}^{\infty} \frac{U_{km}\,\tilde{P}_{kn}}{\frac{\pi^2}{4}(2k-1)^2 + \alpha^2}$$

$$\hat{Q}_{mn} = \sum_{k=1}^{\infty} \frac{U_{km}\,\tilde{Q}_{kn}}{\frac{\pi^2}{4}(2k-1)^2 + \alpha^2} \tag{22}$$

where

$$\tilde{P}_{kn} = \int_{-1}^{1} g_k(z)\, z^2\, S_n''(z)\, dz$$

$$\tilde{Q}_{kn} = \int_{-1}^{1} g_k(z)\, z^2\, S_n(z)\, dz \tag{22a}$$

Equation (21) is already in the form of an eigenvalue problem. Here again the diagonalization is to be performed with help of numerical methods. However we are faced with a new problem. Although we have gone a step further by avoiding the numerical inversion of B, we note that the new matrlx elements appearing in (21) are given in terms of infinite series. Their values will be approximately determined by partially summing them. The results will depend not only upon the number N of functions $S_n(z)$, but also on the number M of functions $g_k(z)$.

We present in the Table 2 some results we obtained by solving (21). We note that if we increase M and N (keeping $M=N$) the convergence toward the right value of R_{cr} is still one sided but slower than in the previous method. We can also keep N constant and let M grow. This pushes down the value of R_{cr}. Increasing N, but holding $N<M$, makes the result come closer to its right values, but now coming from below.

If we compare the results given by the two different procedures, we see that the first are a lot better for the exact values of α_{cr} and R_{cr} were already obtained with relative small matrices. The second method, whlle bringing upper and lower bounds, has a much slower convergence and we should take a lot more functions if the results were

Table 2 - Dependence of the critical Reynolds number upon the N (of functions of the set S_n) and M (of functions of the set g_m). The matrix B was analytically inverted.

M \ N	50	60	70	80	90	100
50	5789					
60		5785				
70	5595		5779			
80		5667		5777		
90	5581				5775	
100		5657	5701	5729		5774
120			5695	5721	5739	
140						5747
150					5733	
180						5743

to converge to a higher degree of accuracy. We may conclude that the error introduced in the numerical inversion of the matrix B are much smaller than those due to the partial sum in the expressions (21).

4. CONCLUSIONS

The solution of the Orr-Sommerfeld equation has been research theme since long time, and several results on this subject have been reported[8-12]. The values of a_{cr} and R_{cr} considered "exact", and that were obtained in the last Section, were first reported by Orszag[4] in 1971. He worked with another spectral method (the τ method) and Chebyshev polynomials as expanding functions. These results have been confirmed by several works[5-7], including this one. A question which has arisen is about how fast can the results converge to the "exact" one, when the number of functions increase.

The functions we used ensure good results with 50 functions and the convergence was tested by taking until 90 functions into the procedure. Moreover the results we got have the property

$$R_{cr}(N = \infty) \leqslant R_{cr}(N) \tag{23}$$

which asserts better and better upper bounds for the exact value of R_{cr}. Chebyshev polynomials present a somewhat faster convergence (about 30 functions) but not one sided as observed in our results. However they do not obey the right boundary conditions of the problem and are not suitable for the use with Galerkin method, whose simplicity and applicability we also wanted to stress in this work.

Another point of interest is the relation of this work with the further development of the stability analysis, now including the effects of the non-linear terrns. The Galerkin expansion we have made constitutes again a good start point for an analysis based on the theory of Lyapunov functions[13]. This has as pre-requisit the knowledge of the solution of the Orr-Sommerfeld equation, as we have determined. Some insight into this problem havealready been gained[7], despite the many extra difficulties, both of numerical as of conceptual nature, that such a non-linear analysis brings with itself.

REFERENCES

1. S. Chandrasekhar, *Hydrodynamic* **and** *Hydromagnetic Stability* - Oxford Press (1961).

2. L.V. Kantorovitch and V. I. Krilov, *Approximate* **Methods** *in Higher Analysis* - P.Noordhoff, Netherland (1958).

3. D..Gottlieb and S.Orszag, *Numerical Analysis* **of** *Spectral* **Methods**- SIAM, Philadelphia (1977).

4. S.Orszag, J. Fluid Mech. 50, 689 (1971).

5. T.Herbert, Die Neutrale Flaeche der ebenen Poiseuille - Stroemung- Habilitationsschrift - Universitaet Sttutgart (1977).

6. J.K.Platten, P.Flandroy and G.Vanderborck, Int. J. Engng. Sci. *12*, 995 (1974).

7. R.F.S.Andrade, Ph.D. Thesis - Universitaet Regensburg (1981).

8. W.Heisemberg, Ann. d. Phys. 74, 577 (1924).
9. C.C. Lin, Quart. Appl. Math. 3, 117, 218, 277 (1945).
10. L.H. Thomas, Phys. Rev. *91*, 780 (1953).
11. S. Shen, J. Aero. Sci. 21, 62 (1964).
12. C.E. Grosh and H. Salwen, J. Fluid Mech. 34, 177 (1968).
13. W. Hahn, *Stability of Motion*, Springer Verlag, Berlin (1967)

RESUMO

Nós usamos o método de Galerkin juntamente com um conjunto de funções ainda não apontadas para resolver a equação de Orr-Sommerfeld e traçar o diagrama de estabilidade do fluxo de Poiseuille bi-dimensional com respeito a perturbações bi-dimensionaismo plano dos parâmetros número de onda a e número de Reynolds R. Os valores obtidos para a_{cr} e R_{cr} estão de acordo com aqueles existentes na literatura.