Revista Brasileira de Física, Volume 13, n? 1, 1983

Coefficients of Fractional Parentage and Interacting S-P Boson Model with Isospin

A. R. SALVETTI

Departamento de Física e Química, Universidade Federal de Mato Grosso do Sul, Caixa Postal, 649, Campo Grande, 79.100, MS, Brasil

and

V. L. MOURA FRANZIN

Departamento de Física Nuclear, Instituto de Física, Universidade de São Paulo, Caixa Postal, 20.516, São Paulo, 01000, SP, Brasil

Recebido em 10 de maio de 1982

Abstract Our purpose with this article is to show how to calculaic analitically the matrix elements of a Harniltonian of a bosons system,using the technique of Coefficients of Fractional Parentage (C.F.P.)(I). The bosons system considered is constitued by isovector s-bosons *(J=0,* T=l) and isoescalar p-bosons (~=1,*T-0)*. We also demonstrate how we can, using such,a technique, reproducethe results for the system of interacting p-bosons **(21,** in the totally symmetric case.

1. INTRODUCTION

Interacting s-p boson model with isospin, which considers the nucleus as being formed by two types of bosons, namely isovector s-bosons (J=0, T=1) and isoscalar p-bosons (J=1, T=0), where J = total angular moment and T = isospin, may have the matrix elements of its Hamiltonian reckoned in an analytical way, through the use of a technique known by the name of Coefficients of Fractional Parentage (C.F.P.). A. Arima and F. lachello[3], in the model of interacting s-d bosons, utilize such a technique; yet, the results obtained remain in the dependence of the calculus of the C.F.P. for the number of bosons to be used sincethe same are not obtained analytically. In our case, we get analytical formulae for the C.F.P. from symmetric states of p-shell and we use such formulae in the calculation of reduced matrix elements; the final results will be a tri-diagonal matrix.

Still as an application of such a method we look upon a system of interacting p-bosons and we secure, for a given interaction, that the energies agree with the results of Chen for the symmetric case.

The group representing our model is group $U_6(JT)$, that may be decomposed as:

$$U_6(JT) \supset U_3(T) \times U_3(J)$$

where the group $U_3(J)$ represents p-bosons system, while the group $U_3(T)$ represents s-bosons system.

2. THE HAMILTONIAN OF THE MODEL

Let the Hamiltonian of one and two bodies be:

$$H = H_1 + H_2 = \sum_{i,j} \langle i|W|j\rangle\, b_i^+ b_j + \frac{1}{2} \sum_{i,j,k,\ell} \langle ij|V_{12}|k\ell\rangle\, b_i^+ b_j^+ b_\ell b_k \qquad (1)$$

Where:

b_i^+ = creation operator for bosons in $\underline{i}$ state

b_i = annihilation operator for bosons in $\underline{i}$ state

In the case of s-p bosons system, we will be able to write the Hamiltonian as:

$$H_1 = \varepsilon_s N_s + \varepsilon_p N_p \qquad (2)$$

$$\begin{aligned} H_2 = {} & 3u_{11} \left[(s^+ \times p^+)^{11} \times (s\times p)^{11}\right]^{00} + \\ & + \sum_T \frac{(2T+1)^{1/2}}{2}\, u_{OT} \left[(s^+\times s^+)^{OT} \times (s\times s)^{OT}\right]^{00} + \\ & + \sum_J \frac{(2J+1)^{1/2}}{2}\, c_{JO} \left[(p^+\times p^+)^{J0} \times (p\times p)^{J0}\right]^{00} + \\ & + \frac{v_{00}}{2} \left\{ \left[(s^+\times s^+)^{00} \times (p\times p)^{00}\right]^{00} + \left[(p^+\times p^+)^{00} \times (s\times s)^{00}\right]^{00} \right\} \end{aligned} \qquad (3)$$

where:

N_s = number of s-bosons

N_p = number of p-bosons

s^+ = creation operator of s-boson

p^+ = creation operator of p-boson

The symbols. $(t_1 \times t_2)^{JT}$ represented the tensorial product of two tensorial operators. The matrix parametrized elements $\langle (j_1 j_2)\ (t_1\ t_2),\ JT\ |\ V_{12}|(j_1' j_2')(t_1' t_2'),\ JT\rangle$ are given by:

$$u_{0T} = \langle (00)(11),0T|V_{12}|(00)(11),0T\rangle = \langle ss,0T|V_{12}|ss,0T\rangle$$

$$c_{J0} = \langle (11)(00),J0|V_{12}|(11)(00),J0\rangle = \langle pp,J0|V_{12}|pp,J0\rangle$$

$$v_{00} = \langle (00)(11),00|V_{12}|(11)(00),00\rangle = \langle pp,00|V_{12}|pp,00\rangle$$

$$u_{11} = \langle (01)(10),11|V_{12}|(01)(10),11\rangle = \langle sp,11|V_{12}|sp,11\rangle$$

We will use the basis $U_3(T) \times U_3(J)$ to represent the states of our Hamiltonian; in that basis the number of s-bosons and the number of p-bosons are well defined. The notation used will be:

$$|N_s N_p\rangle = |N_s(J_s M_s T_s M_{T_s})N_p(J_p M_p T_p M_{T_p});J\ M\ T\ M_T\rangle$$

3. THE METHOD OF COEFFICIENTS OF FRACTIONAL PORCENTAGE (C.F.P.)

To calculate the matrix elements of our bosons system, we'll use the method of C.F.P.. In this method we consider a particle of the particle system, separated of the others, and the symmetry required is satisfied by the calculation of the configuration functions, asa linear combination of the functions vectorially coupled of that separate particle and the permitted states of the other particles. The coefficients which determine that particular linear combination are called C.F.P..

$$|j^N \alpha JM\rangle = \sum_{\alpha_1, J_1} \left[j^{N-1}(\alpha_1 J_1) jJ | j^N \alpha J\right] \quad |j^{N-1}_{1\ldots N-1}(\alpha_1 J_1)\ j_N\ J\ M\rangle \qquad (4)$$

where

$$\left[j^{N-1}(\alpha_1 J_1) jJ\ |\ j^N \alpha J\right] = \text{C.F.P.}$$

$\underline{\alpha}$ and $\underline{\alpha}_1$ are additional quantum numbers, which are necessary if there are several states with the same value of $\underline{J}$.

In the literature we find relations of recurrency for calculation of the C.F.P.; yet, we are interested in an analytic formula for the C.F.P. of the states:

$$|p^N(N)J\ M> \ ; \quad |s^N(N)T\ M_T>$$

where (N) labels an entirely symmetric state in the N particle (bosons) composing it. For procuring the desired analytical formulae, we use the recurrency relations derived by Shalit and Talmi[4] in the seniority scheme, making seniority equal to the angular moment, after converting the result for the totally symmetric case. At that rate we obtain:

$$\left[p^N(N)J|p^{N-1}(N-1)J-1\right] = \left[\frac{J(N+J+1)}{(2J+1)N}\right]^{1/2} \tag{5}$$

$$\left[p^N(N)J|p^{N-1}(N-1)J+1\right] = \left[\frac{(J+1)(N-J)}{(2J+1)N}\right]^{1/2} \tag{6}$$

From the formulae (5) and (6) and using the recurrency relation[5],

$$\left[p^N(a)J|p^{N-2}(a')J',p^2(a'')J''\right] = \sum_{a_1,J_1} \left[p,p|p^2(a'')J''\right] .$$

$$.\ \left[p^{N-1}(a_1)J_1|p^{N-2}(a')J',p\right]\left[p^N(a)J|\ p^{N-1}(a_1)J_1,p\right] .$$

$$.\ \langle J',pp(J'');J|J'p(J_1)p;J\rangle \tag{7}$$

where

$$\langle J',pp(J'');J|J'p(J_1)p;J\rangle = (-1)^{2p+J'+J}\ U(J'111;J_1J'')$$

with:

$$U(j_1j_2J1;J_{12}J_{23}) = \begin{Bmatrix} j_1 & j_2 & J_{12} \\ 1 & J & J \end{Bmatrix} .\ \left((2J_{12}+1)(2J_{23}+1)\right)^{1/2}$$

we secure the C.F.P. for the simultaneous separation of two symmetric particles, as follows:

$$\left[p^N(N)J|p^{N-2}(N-2)J,p^2S\right] = \left[\frac{(N-J)(N+J+1)}{3N(N-1)}\right]^{1/2} \tag{8}$$

$$\left[p^N(N)J|p^{N-2}(N-2)J-2,p^2D\right] = \left[\frac{J(J-1)(N+J+1)(N+J-1)}{(2J+1)(2J-1)N(N-1)}\right]^{1/2} \tag{9}$$

$$\left[p^N(N)J|p^{N-2}(N-2)J,p^2D\right] = \left[\frac{2J(J+1)(N-J)(N+J+1)}{3(2J-1)(2J+3)N(N-1)}\right]^{1/2} \tag{10}$$

$$\left[p^{N}(N)J|p^{N-2}(N-2)J+2,p^{2}D\right] = \left[\frac{(J+1)(J+2(N-J)(N-J-2))}{(2J+1)\ (2J+3)N(N-1)}\right]^{1/2} \tag{11}$$

Using the C.F.P. we can calculate matrix reduced elements in a compact form, for:

$$\langle N_p+1,J'\|\,p^{+}\|\,N_p,J\rangle = ((N_p+1)(2J'+1))^{1/2}\left[p^{N_p}(N_p)J\,|\,p^{N_p+1}(N+1)J'\right] \tag{12}$$

$$\langle N_p,J\|\,p\,\|\,N_p+1,J'\rangle = (-1)^{J-J'}\langle N_p+1,J'\|\,p^{+}\|\,N_p,J\rangle \tag{13}$$

$$\langle N_p+2,J'\|\,(p^{+}\times p^{+})^{L}\|\,N_p,J\rangle = ((N_p+1)(N_p+2)(2J'+1))^{1/2}$$

$$\left[p^{N_p}(N_p)J,p^{2}L|p^{N_p+2}\ (N_p+2)J'\right] \tag{14}$$

In the case of s-bosons the calculation is identical, being enough to change N_p by N_s, J by T and $p(p^{+})$ by $s(s^{+})$.

4. THE MATRIX ELEMENTS

With the C.F.P. obtained, we can show that:

$$\langle N_s'\,N_p'\ |H|N_sN_p\rangle = \Big[\varepsilon_sN_s + \varepsilon_pN_p + N_sN_p\,u_{11} + \frac{u_{00}}{6}\,(N_s-T)(N_s+T+1) +$$

$$+\frac{u_{02}}{6}\,(T(T+1)+2N_s(N_s-2)) + \frac{c_{00}}{6}\,(N_p-J)(N_p+J+1) +$$

$$+\frac{c_{20}}{6}\,(J(J+1)+2N_p(N_p-2)\Big]\ \delta_{N_s'N_s}\cdot\delta_{N_p'N_p} +$$

$$+\frac{v_{00}}{6}\Big[((N_s-T+2)(N_s+T+3)(N_p-J)(N_p+J+1))^{1/2}\ \delta_{N_s'\,N_s+2}\cdot\delta_{N_p'N_p-2} +$$

$$+((N_s-T)(N_s+T+1)(N_p-J+2)(N_p+J+3))^{1/2}\ \delta_{N_s'\,N_s-2}\cdot\delta_{N_p'\,N_p+2}\Big] \tag{15}$$

5. INTERACTING P-BOSON SYSTEM

Let's consider a system formed exclusively by interacting p--bosons. Let's calculate the energy of that system for an interaction of the type:

$$V = \sum_{i<j} V_{ij} \quad \text{where} \quad V_{ij} = a + b\vec{j}_i \cdot \vec{j}_j \tag{16}$$

In the representation of the occupation number, that interaction is written as:

$$V = \frac{1}{2} \sum_{i,j,k,\ell} \langle ij|V_{12}|k\ell\rangle \; b_i^+ b_j^+ b_\ell b_k$$

In the case of a p-bosons system, we have:

$$V = \sum_J \langle pp,J0|V_{12}|pp,J0\rangle \frac{(2J+1)^{1/2}}{2} \left[(p^+ \times p^+)^{J0} \times \cdot (p \times p)^{J0}\right]^{00} \tag{17}$$

By using the previous relation and the formulae obtained for the C.F.P., we can show that:

$$\langle N_p'|V|N_p\rangle = E_p \, \delta_{N_p'N_p} = \left[N_p(N_p-1)\frac{a}{2} + (J(J+1)-2N_p)\frac{b}{2}\right] \delta_{N_p'N_p} \tag{18}$$

REFERENCES

1. G.Racah, Phys. Rev. 63, 367 (1943); G. Racah, Phys. Rev. 76, 1352 (1949).
2. C.H.T. Chen, IFUSP Preprint IFUSP/P-235, 1981.
3. A. Arima and F. Iachello, Annals of Physics 99, 253-317 (1976). A. Arima and F. Iachello, Annals of Physics *111*, 201-238 (1977).
4. A. de-Shalit and I. Talmi, ***Nuclear Shell Theory***, (Academic, New York, 1963).
5. H. Horie, Journal of the Physical Society of Japan 19, 1782 (1964).

RESUMO

Nosso objetivo com este artigo, é mostrar como se calcula analiticamente os elementos de matriz de uma hamiltoniana de um sistema de bósons, usando a técnica dos Coefficients of Fractional Parentage (c.F.P.) (1). O sistema de bósons considerado, é constituido por bõsons isovetoriais s (J=0, T=1) e bõsons isoescalares p (J=1, T=0). Mostramos também, como podemos, usando tal técnica, reproduzir os resultados para o sistema de bõsons p interatuantes (2), no caso totalmente simétrico.