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Ordering Operator Technique Applied to Open Systems

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Abstract We use the normal ordering technique and the coherent representation to describe the evolution of an open system of a single oscillator, linearly coupled with an infinite number of reservoir oscillators, and we show how to include the dissipation and obtain the exponential decay.

1. INTRODUCTION

In recent years the dynamics of an open system S coupled to a reservoir R has received considerable attention[1-4]. According to some authors[1], the open systems are of greater importance than isolated ones. One of the first derivations of an equation of motion for an open system was given by Bergmann and Lebowitz[5] and, after them, several authors have treated this question in different approaches[6-8]. The problems which emerge when treating open systems usually refer to: (a) the initial states and its influence on the solution; (b) the problem of the ergodicity; (c) the Markoffian character of the equation of motion; (d) the construction of exactly solvable models; (e) the different techniques employed to derive the equation of motion; etc.

The basic starting point for discussion of an open system is the density matrix $\rho(t)$ for the composite system $S+R$. The density matrix obeys the Liouville equation and the behaviour of the open system S is inferred from the reduced density matrix $\rho_S(t) = tr_R\,[\rho_{SR}(t)]$, where the variables of R have been eliminated.

In this paper we consider this problem by first treating an open system in a pure state. This is possible by setting the reservoir

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R at zero temperature or even in a coherent state. The main purpose of the present work is to display a different and simple technique inwhich we use the normal order character of the Hamiltonian H and of thetime evoiution operator $U(t)$.

The outline of the technique can be surnmarized in the following way: given the Schrödinger equation ($\hbar = 1$)

$$i \frac{\partial|\psi(t)>}{\partial t} = H|\psi(t)> . \qquad (1)$$

where $H = H(a,a^{+})$, $a^{+}(a)$ is the creation (annihilation) operator for some particle, $|\psi(t)>$ stands for the whole system S+R, and setting

$$|\psi(t)> = U(t) \ |\psi(0)> \ ; \quad U(0) = 1 \ , \qquad (2)$$

we obtain

$$i \frac{\partial U}{\partial t} = HU \ . \qquad (3)$$

If $H(a,a^{+})$ is in normal order, which means that all the creation operators a^{+} are on the left of all annihilation operators a,

$$H(a,a^{+}) = \sum_{m,n} h_{mn}(t)\,(a^{+})^{m}(a)^{n} \qquad (4)$$

and if we assume that $U(t)$ is also in normal order, which is written as $U^{(N)}(t)$, the right-hand side of (4) may be set in normal or-der, by using the equality[9]

$$a^{n}U^{(N)} = N\left[(\alpha + \frac{\partial}{\partial\bar{\alpha}})^{n}\ \bar{U}^{(N)}(\alpha,\bar{\alpha},t)\right] \qquad (5)$$

where $\bar{U}^{(N)}(\alpha,\bar{\alpha},t) = <\alpha|U^{(N)}(t)|\alpha>$, $|\alpha>$ is a coherent state, i.e., $a|\alpha>$ $= \alpha|\alpha>$ and **N** is the normal ordering operator[9].

In this way, insertinq (4) into (3) and using (5), we get

$$i \frac{\partial\bar{U}^{(N)}}{\partial t} = \sum_{m,n} h_{mn}(t)\,(\bar{\alpha})^{m}\ (\alpha + \frac{\partial}{\partial\bar{\alpha}})^{n}\ \bar{U}^{(N)} \qquad (6)$$

which is an equation for $\bar{U}^{(N)}(\alpha,\bar{\alpha},t)$. When (6) is solved we obtain $|\psi(t)>$ by

$$|\psi(t)> = N\ \{\bar{U}^{(N)}(\alpha,\bar{\alpha},t)\}\ |\psi(0)> \quad . \tag{7}$$

The advantage of this method is to transform an operator equation (see (3)) in a c-number equation (see (6)).

2. A SIMPLE EXAMPLE

As a simple example we solve the Schrödinger equation for the case of two weakly coupled oscillators A and B described by the Hamiltonian

$$H = \omega a^{\dagger}a + \Omega b^{\dagger}b + \gamma(ab^{\dagger} + a^{\dagger}b) \quad , \tag{8}$$

where the interaction term (last term in (8)) is assumed to be of the type introduced by Gordon et al[10]; ω and R are, respectively, the corresponding frequencies of the oscillators A and B and γ is the coupling constant . As it can be seen, the Hamiltonian (8) is in normal order. On assuming that U is also in normal order, we get

$$i\frac{\partial U^{(N)}}{\partial t} = \omega a^{\dagger}\left(U^{(N)}a + \frac{\partial U^{(N)}}{\partial a^{\dagger}}\right) + \Omega b^{\dagger}\left(U^{(N)}b + \frac{\partial U^{(N)}}{\partial b^{\dagger}}\right)$$

$$+ \gamma a^{\dagger}\left(U^{(N)}b + \frac{\partial U^{(N)}}{\partial b^{\dagger}}\right) + \gamma b^{\dagger}\left(U^{(N)}a + \frac{\partial U^{(N)}}{\partial a^{\dagger}}\right) \quad . \tag{9}$$

At this point we use the coherent representation. Setting

$$\bar{U}^{(N)}(\alpha,\bar{\alpha},\beta,\bar{\beta},t) = <\alpha,\beta|U^{(N)}|\alpha,\beta> \quad , \tag{10}$$

where $|\alpha,\beta>$ is such that

$$a|\alpha,\beta> = \alpha|\alpha,\beta>$$

$$b|\alpha,\beta\rangle = \beta|\alpha,\beta\rangle \quad , \tag{11}$$

and putting

$$\bar{U}^{(N)} = \exp\left[G(\alpha,\bar{\alpha},\beta,\bar{\beta},t)\right] \quad , \tag{12}$$

where

$$G = A(t)\bar{\alpha}\alpha + B(t)\bar{\beta}\beta + C(t)\bar{\alpha}\beta + D(t)\alpha\bar{\beta} \ , \tag{13}$$

we find two systems of coupled equations

$$\begin{aligned} i\dot{\mathcal{A}} &= \omega\mathcal{A} + \gamma D \\ i\dot{D} &= \Omega D + \gamma\mathcal{A} \end{aligned} \tag{14}$$

and

$$\begin{aligned} i\dot{\mathcal{B}} &= \Omega\mathcal{B} + \gamma C \\ i\dot{C} &= \omega C + \gamma\mathcal{B} \ , \end{aligned} \tag{15}$$

where $\mathcal{A}(t) = A(t) + 1$; $\mathcal{B}(t) = B(t) + 1$ and (cf. (2)) $A(0) = B(0) = C(0) = D(0) = 0$. Solving (14) and (15) we find

$$\mathcal{A}(t) = \exp(-i(\omega+\Omega)t/2)\left[\cos\Gamma t - i\left(\frac{\omega-\Omega}{2\Gamma}\right)\sin\Gamma t\right] \tag{16}$$

$$D(t) = -i(\gamma/\Gamma)\exp\left[-i(\omega+\Omega)t/2\right]\sin\Gamma t \tag{17}$$

$$\mathcal{B}(t) = \mathcal{A}(t) \quad (\omega \leftrightarrow \Omega) \ ; \ C(t) = D(t) \quad (\omega \leftrightarrow \Omega) \ , \tag{18}$$

where

$$\Gamma = \left[\left(\frac{\omega-\Omega}{2}\right)^2 + \gamma^2\right]^{1/2} \quad . \tag{19}$$

Inserting these results (they should be compared with those in ref. 9, pag. 207) into (12) and (13) and using (7), we obtain $|\psi(t)>$ in terms of the initial state $|\psi(0)>$. If we suppose the initial state is a coherent state, i.e., $|\psi(0)> = |\alpha,\beta>$, we find

$$|\psi(t)> = e^{(A\alpha+C\beta)a^{\dagger} + (B\beta+D\alpha)b^{\dagger}}|\alpha,\beta> , \qquad (20)$$

which shows that a initially coherent state remains a coherent state. This occurs because the present Hamiltonian (see (8)) belongs to the class of Hamiltonians that mantains the coherence of an initially coherent state[11].

If we assume that $\beta=0$, which means that an oscillator is in the vacuum state, then

$$|\psi(t)> = e^{\alpha(Aa^{\dagger} + Db^{\dagger})}|\alpha,0> . \qquad (21)$$

This shows that only $A(t)$ and $D(t)$ are relevant for the solution when the oscillator B is into the vacuum state. It is also interesting to note that only $D(t)$ is coupled to $A(t)$ in (14).

3. GENERALIZATION TO N OSCILLATORS

Let us suppose now we have one oscillator A interacting with two oscillators B_1 and B_2 (we neglect the interaction between B_1 and B_2). In this case the Hamiltonian is

$$H = \omega a^{\dagger}a + \Omega_1 b_1^{\dagger}b_1 + \Omega_2 b_2^{\dagger}b_2 + \gamma_1(a^{\dagger}b_1 + ab_1^{\dagger}) + \gamma_2(a^{\dagger}b_2 + ab_2^{\dagger}) . \qquad (22)$$

Thus, according to the foregoing example, we set

$$\bar{U}^{(N)} = \exp[G(\alpha,\bar{\alpha},\beta_1,\bar{\beta}_1,\bar{\beta}_2,\beta_2,t)] , \qquad (23)$$

where

$$G = A\bar{\alpha}\alpha + B_1\bar{\beta}_1\beta_1 + B_2\bar{\beta}_2\beta_2 + C_1\bar{\alpha}\beta_1 + D_1\alpha\bar{\beta}_1 + C_2\bar{\alpha}\beta_2 + D_2\alpha\bar{\beta}_2 \qquad (24)$$

and we obtain, after some algebraical procedure,

$$i\dot{A} = \omega A + \gamma_1 D_1 + \gamma_2 D_2$$
$$i\dot{D}_k = \Omega_k D_k + \gamma_k A \; ; \quad k = 1,2 \; . \qquad (25)$$

Here, we are not interested in the functions $B_k(t)$ and $C_k(t)$. The generalization for the case of one oscillator A interacting with N oscillators is straightforward and leads to the system of coupled equations

$$i\dot{A} = \omega A + \sum_{k=1}^{N} \gamma_k D_k$$
$$i\dot{D}_k = \Omega_k D_k + \gamma_k A \; . \qquad (26)$$

4. THE CASE N→∞ AND IRREVERSIBILITY

On assuming now that $N\to\infty$, Ω_k will belong to a continuous spectrum[12]. Once more, according to the foregoing procedures, we set

$$G = G(\alpha,\bar{\alpha},\{\beta_k\},\{\bar{\beta}_k\}, t)$$
$$= A\bar{\alpha}\alpha + \int B_k\bar{\beta}_k\beta_k dk \qquad (27)$$
$$+ \bar{\alpha} \int C_k\beta_k dk + \alpha \int D_k\bar{\beta}_k dk \; ,$$

and we obtain

$$i\dot{A} = \omega A + \int \gamma_k D_k dk$$
$$i\dot{D}_k = \Omega_k D_k + \gamma_k A \tag{28}$$

In order to solve this system of coupled equations we introduce the Laplace transform

$$\tilde{A}(p) = \int_0^\infty dt\, e^{-pt} A(t) \ . \tag{29}$$

By eliminating D_k in (28) we find

$$\tilde{A}(p) = \left[p + i\omega - i\int_0^\infty \left(\frac{\gamma_k^2}{\Omega_k - ip}\right) dk\right]^{-1} , \tag{30}$$

which may be inverted, yieiding

$$A(t) = \frac{1}{2\pi i}\int_{\lambda-i\infty}^{\lambda+i\infty} dp\, e^{pt}\left[p + i\omega - i\int_0^\infty \left(\frac{\gamma_k^2}{\Omega_k - ip}\right) dk\right]^{-1} \tag{31}$$

where λ is the abcissa of converaence associated to the Laplace transform. The preceding result is similar to those obtained byother methods (see refs. 1 and 2). Thus, if we make the assumption of weakly coupled oscillators in such a way that γ_k is a smooth function in the whole range of integration and assuming

$$\int_0^\infty \frac{\gamma_k^2}{\Omega_k}\, dk < \omega \tag{32}$$

holds, the integrand in (31) is holomorphic in the entire complex plane except on the positive real axis. A deformation of the path of integration such that it encircles the positive real axis yields (see, e.g., ref. 1)

$$A(t) = \frac{1}{\pi}\int_0^\infty \frac{dx\, e^{-ixt}\, J(x)}{[\omega - x - \Delta(x)]^2 + J^2(x)} , \tag{33}$$

where

$$J(x) = \pi\gamma^2(x)$$
$$\Delta(x) = P\int_0^\infty \frac{\gamma^2(x)}{\Omega_k - x}\, dk \quad . \tag{34}$$

In order that assumption of weakly coupled oscillators make sense we assume that

$$\Delta(x), \quad J(x) << \omega \quad \forall\ x \ . \tag{35}$$

In addition we assume that $\Delta(x)$ and $J(x)$ are slowly varying functions around $x=\omega$, otherwise the interaction would vary strongly in the neighbourhood of w. Thus we may make the approximations

$$\Delta(x) \simeq \Delta(\omega)$$
$$J(x) \simeq J(\omega) \tag{36}$$

The denominator in (33) has one zero in the lower half plane given by

$$x = \omega - \Delta(\omega) - iJ(\omega) \ , \tag{37}$$

which leads (33) to

$$A(t) = e^{-i\tilde{\omega}t - J(\omega)t} \tag{38}$$

where $\tilde{\omega} = \omega - \Delta(\omega)$, which shows a frequency-shift in ω due to the weak--coupling among the oscillator A and the oscillators B_k. Golng back to (7) and putting $|\psi(0)\rangle = |\alpha, 0\rangle = |\alpha\rangle|0\rangle$, we find

$$|\psi(t)\rangle = \left[e^{A(t)\alpha a^\dagger}|\alpha\rangle \left[e^{\alpha\int D_k(t) b_k^\dagger dk}|0\rangle\right]\right. \ . \tag{39}$$

This result shows that, except for a normalization factor, we may write the (pure) state for the subsystem A as

$$|\psi_A(t)\rangle = e^{A(t)\alpha a^\dagger}|\alpha\rangle \tag{40}$$

It is easy to show that

$$a|\psi_A(t)\rangle = \alpha A(t)\,|\psi_A(t)\rangle \ , \tag{41}$$

and $|\psi_A(t)\rangle$ is an eigenvector of the annihilation operator a with eigen-value given by

$$\alpha'(t) = \alpha A(t) = \alpha\ e^{-i\tilde{\omega}t - J(\omega)t} \tag{42}$$

The foregoing result shows explicitly the dissipation: the vector representing $\alpha'(t)$ in the complex plane rotates clockwise with the shifted frequency $\tilde{\omega}$ and has an exponential decay with a lifetime $\tau = J^{-1}(\omega)$. The coherent character of the state $|\psi_A(t)\rangle$ is due to the form of the Hamiltonian and also to the initial state (reservoir at zero temperature).

5. COMMENTS AND CONCLUSION

The precedent description refers to an open system of a single oscillator coupled to N oscillators $(N\to\infty)$, but the formal analogy between an oscillator and the electromagnetic field could also easily be transposed to the problem of an open system of a quantized (single mode) free radiation field. This is just the case of the electromagnetic field in the single mode laser theory.

An apparent question that emerges from the present treatment is: how to generalize this technique in order to describe an open system in a mixed state? In this case, as it is well known, we substitute $|\psi(t)\rangle \to \rho(t)$ and $U(t) = e^{-iHt} \to U(t) = e^{-itL}$, where L is the Liouville operator, viz., $L\rho = [H,\rho] = H\rho - \rho H$. Then (3) transforms into $i\partial U/\partial t =$ $= LU = HU - UH$.

Therefore, whereas in the pure state case H is an explicit operator in terms of a, a^{+}, b_k and b_k^{+}, in the mixed state case, with L, the same does not accur. Then the generalization of the normal order technique to mixed states seems to be not trivial. This question has also been investigated by us and will be published elsewhere.

NOTES AND REFERENCES

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12. Similar procedure can be found in Weisskopf - Wigner theory of natural linewidth in atoms (see, e.g. ref, 9, pag. 285).

RESUMO

Usamos a técnica de ordenação de operadores e a representação coerente para descrever a evolução temporal de um sistema aberto - constituído de um oscilador acoplado a um sistema infinito de osciladores de um reservatório de perdas - e mostramos como incluir a dissipação e obter o decaimento exponencial.