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Gravitational Gauge Approach for the SO(4,1) Group

E. STÉDILE

Departamento de Física, Universidade Federal do Paraná, Curitiba, 80.000, PR, Brasil

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Abstract A sourceless gauge theory for gravitation is examined. By considering the de Sitter group in a bundle of frames, Yang and Einstein equations are derived from a Yang-Mills classical theory. The approach proposed has a gauge-like action integral wich leads to the field equations.

1. INTRODUCTION

Recent advances of gauge theories for electro-weak interactions and the promising approach of chromodynamics to strong processes have put forward expectations that also gravitation will soon, leave its isolation and have a gauge formulation, too. At the classical level the analogies between Yang-Mills (YM) theory and General Relativity (GR), from the geometrical point of view, have been noticed since long[1]. However, the Hilbert-Einstein Lagrangian is not of the YM-type, and the dynamical aspects of the two theories are qualitatively different. Also, gauge theories involve groups acting on interna1 spaces while gravitation is concerned with space-time itself. That gravitation is more intimately connected to space-time comes from soldering, a propertyofthe bundle of frames, which is absent in the bundles lying behind the usual gauge theories[2]. It is related to theaffine character of the tangent spaces and to the torsion, and shows itself in any differentiable manifold. Its main consequencesare an extra Bianchi identity and, if duality symmetry is to remain valid, an extra YM equation[3].

2. GEOMETRICAL CONSIDERATIONS

Our objective is to obtain a theory for the de Sitter group SO(4,1) with all the essential characteristics of a gauge theory. The formalism is developped in a bundle of frames $P(M,G)$, M being the Minkowski spaca and G the symmetry group. Gauge transformations are interpreted as changes of bases along the fiber space. We will use here-after the compact notation of differential forms.

The Lie algebra G' of the gauge group is

$$[J_a, J_b] = f^c_{ab} \, J_c \tag{2.1}$$

where f^c_{ab} are structure constantsof G. A gauge potential is a 1 form A (connection form) valued in G': given a basis $\{J_a\}$ of generators in G' and a coordinate system $\{x^\mu\}$, we have

$$A = J_a \, A^a{}_\mu \, dx^\mu \, , \tag{2.2}$$

A^a being the usual gauge potentials. The gauge field strengh is the curvature of A

$$F = dA + A \wedge A \, . \tag{2.3}$$

It is a 2 form in the adjoint representation of G, and in the particular system of coordinates above it has the cornponents $F^a_{b\mu}$ given by

$$F = \frac{1}{2} J_a \, F^a{}_{\mu\nu} \, dx^\mu \wedge dx^\nu =$$

$$= \frac{1}{2} J_a \, (\partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f^a{}_{bc} A^b_\mu A^c_\nu) \, dx^\mu \wedge dx^\nu \, . \tag{2.4}$$

Bianchi identity comes by differentiation of (2.3):

$$DF = dF + [A, F] = 0 \tag{2.5}$$

setting that the covariant derivative of F is automatically zero. However, all gauge theories exhibit duality syrnmetry, which says that (for the sourceless case) YM equations are just (2.5) written for the dual of F:

$$D^*F = d^*F + [A, {}^*F] = 0 \, . \tag{2.6}$$

In the presence of sources, D^*F is equal to the Noether current densities whose corresponding charges generate the gauge group. When a source current is present in (2.6) we rnight be tempted to add convenient sources to (2.5) in order to preserve duality. However, in this case, (2.3) fails to be true everywhere. For this reason we prefer to adopt the point of view that *duality is a symmetry of the sourceless case broken by the source currents.*

Now, equations (2.5) and (2.6) have very different origins. The former is an identity of purely geometrical content, and the latter is a physical equation, resulting from the choice of the invariant action

$$I = -\frac{1}{4}\int \mathrm{tr}\ (F \wedge {}^*F)\ . \tag{2.7}$$

3. GAUGE FIELD FOR THE AFFINE GROUP

Before considering the $SO(4,1)$ group itself, we show an approach to the affine group $AL(4,R) = GL(4,R) \odot T_4$. Here $GL(4,R)$ is the linear group and T_4 the translational group. For the algebra $GL'(4,R)$ of the former we take the canonical basis $\{\Delta^a{}_b\}$ whose matrices obey the Lie algebra commutation rules

$$[\Delta^a{}_b, \Delta^c{}_d] = (\delta^c{}_b\ \delta^m{}_d\ \delta^a{}_n - \delta^a{}_d\ \delta^m{}_b \delta^c{}_n) \Delta^n{}_m\ , \tag{3.1}$$

and for the latter

$$[I_a, I_b] = 0\ . \tag{3.2}$$

Since the Lie algebra $AL'(4,R)$ is a vector space and the direct summation $GL'(4,R) \oplus R^4$, we can define a connection $\bar{\Gamma}$ (gauge potencial of the affine group) by

$$\bar{\Gamma} = \Gamma + S \tag{3.3}$$

where Γ is referred to the rotational part of $AL(4,R)$ (i.e. $GL(4,R)$) and S to its translational part. So, Γ may be interpreted as a gauge potential of the rotational sector of $AL(4,R)$, given by

$$\Gamma = \Delta^b{}_a\ \Delta^a{}_{b\mu} dx^\mu \tag{3.4}$$

and S as a gauge potential of the translational sector

$$S = I_a \, h^a_{\ \mu} \, dx^\mu \ . \tag{3.5}$$

Geometrically S is the solder-form, valued on R^4 and h^a_μ play the role of four-legs.

The gauge field is now

$$\bar{F} = d\bar{\Gamma} + \bar{\Gamma} \wedge \bar{\Gamma} = F + T \ , \tag{3.6}$$

where

$$F = d\Gamma + \Gamma \wedge \Gamma \ , \tag{3.7}$$

is the field (curvature) referred to the rotational sector, and

$$T = dS + \Gamma \wedge S + S \wedge \Gamma \tag{3.8}$$

the field (torsion) referred to the translational sector. These fields satisfy Biachi identities

$$dF + [\Gamma, F] = 0 \tag{3.9}$$

and

$$dT + [\Gamma, T] + [S, F] = 0 \ . \tag{3.10}$$

Further-more, the total gauge field $\bar{F}$ satisfies Bianchi identity

$$d\bar{F} + [\bar{\Gamma}, \bar{F}] = 0 \tag{3.11}$$

which may be decomposed into (3.9) and (3.10).

If we consider now duality simmetry, YM equation is the same as (3.11) but written for $*\bar{F}$:

$$d{*\bar{F}} + [\bar{\Gamma}, *\bar{F}] = 0 \ . \tag{3.12}$$

Such equation may be decomposed into

$$d{*F} + [\Gamma, {*F}] = 0 \, , \tag{3.13}$$

and

$$d{*T} + [\Gamma, {*T}] + [S, {*F}] = 0, \tag{3.14}$$

for the two sectors. These equations have been proposed by Popov and Daikhin[4] based on heuristic arguments. In addition, they have pointed out that for a Levi-Civita connection (3.13) reduces to Yang's equation[5]

$$R_{\mu\nu;\lambda} - R_{\mu\lambda;\nu} = 0 \, . \tag{3.15}$$

Also, (3.14) in cornponent form is

$$\partial^{\lambda} T^{a}_{\mu\lambda} + \Gamma^{a}_{b\lambda} T^{b}{}_{\mu}{}^{\lambda} + S^{b}_{\lambda} F^{a}{}_{b}{}_{\mu}{}^{\lambda} = 0 \, , \tag{3.16}$$

which stablishes propagation for the torsion field. In the torsionless case it reduces to Einstein free equation[3]

$$R_{\mu\nu} = 0 \, . \tag{3.17}$$

4. THE SO(4,1) THEORY

Since SO(4,1) is an orthogonal group, the total gauge field

$$\tilde{F} = J_{ab} \tilde{F}^{ab} \tag{4.1}$$

has the representation

$$\tilde{F} = \begin{bmatrix} 0 & \tilde{F}^{12} & \tilde{F}^{13} & \tilde{F}^{14} & \tilde{F}^{15} \\ \tilde{F}^{21} & 0 & \tilde{F}^{23} & \tilde{F}^{24} & \tilde{F}^{25} \\ \tilde{F}^{31} & \tilde{F}^{32} & 0 & \tilde{F}^{34} & \tilde{F}^{35} \\ \tilde{F}^{41} & \tilde{F}^{42} & \tilde{F}^{43} & 0 & \tilde{F}^{45} \\ \tilde{F}^{51} & \tilde{F}^{52} & \tilde{F}^{53} & \tilde{F}^{54} & 0 \end{bmatrix} \tag{4.2}$$

in the algebra SO'(4,1).

In (4.1) each generator is given by

$$J_{ab} = \begin{bmatrix} 0 & \cdots & \cdots & -1 & \cdots \\ \vdots & & & & \\ \vdots & & & & \\ 0 & \cdots & & +1 & \cdots \end{bmatrix} \qquad (4.3)$$

for the a-th row b-th column. The commutation rules for these generators are

$$[J_{ab}, J_{cd}] = \frac{1}{2} (\delta_{ab} J_{cd} - \delta_{ac} J_{bd} + \delta_{bc} J_{ad} - \delta_{bd} J_{ac}) \qquad (4.4)$$

with $a,b,c,\ldots = 1,2,\ldots 5$. Now we separate these 10 generators in the following way: we choose 6 of them such that they satisfy Lorentz algebra, and define the others by

$$\Pi_\mu = \frac{1}{\alpha} J_{5\mu} \qquad (\mu = 1, 2, 3, 4) \qquad (4.5)$$

a being a parameter. In this way (4.4) may be decomposed into

$$\begin{aligned} [J_{\sigma\lambda}, J_{\mu\nu}] &= \frac{1}{2} (\delta_{\sigma\nu} J_{\lambda\mu} - \delta_{\sigma\mu} J_{\lambda\nu} + \delta_{\lambda\mu} J_{\sigma\nu} - \delta_{\lambda\nu} J_{\sigma\mu}) , \\ [\Pi_\lambda, J_{\mu\nu}] &= \frac{1}{2} (\delta_{\lambda\mu} \Pi_\nu - \delta_{\lambda\nu} \Pi_\mu) , \qquad (4.6) \\ [\Pi_\mu, \Pi_\nu] &= - \frac{1}{2\alpha^2} J_{\mu\nu} . \end{aligned}$$

This shows that the generators Π do not commute with themself, unless we take the limit $\alpha \to \infty$. In thiscase the 10 generators above will satisfy the Poincaré algebra. We can interpret a as the de Sitter sphere radiusand $\alpha \to \infty$ corresponds to the transformation of such a sphere in a 4-dimensional plane. Rotations along the planes (x_μ, x_5) will become translations in space-time[6]. In the limit $\alpha \to \infty$, the generators Π will be translations generators

$$\lim_{\alpha\to\infty} \Pi_\mu = P_\mu \quad , \tag{4.7}$$

P_μ being the energy-momentum operators defined in space-time.

Following last section the gauge potential is

$$\tilde{\Gamma} = \Gamma + S \, , \tag{4.8}$$

where

$$\Gamma = J_a{}^b \, \Gamma^a{}_{b\mu} dx^\mu \tag{4.9}$$

and

$$S = \Pi_a \, S^a{}_\mu \, dx^\mu \tag{4.10}$$

with $a,b,c,\ldots = 1,2,3,4$. If we take now YM equation for the gauge field $\tilde{F} = d\tilde{\Gamma} + \tilde{\Gamma} \wedge \tilde{\Gamma}$, that is

$$d{*}\tilde{F} + [\tilde{\Gamma}, {*}\tilde{F}] = 0 \, , \tag{4.11}$$

we will have, after separation in the two sectors,

$$d{*}F + [\Gamma, {*}F] + [S, {*}T] = 0 \, , \tag{4.12}$$

$$d{*}T + [\Gamma, {*}T] + [S, {*}F] = 0. \tag{4.13}$$

But (4.13) is the same as (3.14) which reduces to Einstein's equation (3.17), while (4.12) has the additional term $[S, {*}T]$ if compared to (3.13). In components this equation is

$$\partial^\nu F^a{}_{b\mu\nu} + \Gamma^a{}_c{}^\nu F^c{}_{b\mu\nu} - \Gamma^c{}_b{}^\nu F^a{}_{c\mu\nu} - \frac{1}{2\alpha^2} S^a{}_\lambda T_b{}^{\mu\lambda} = 0 \, . \tag{4.14}$$

If we take the limit $\alpha\to\infty$ and consider a Levi-Civita connection, (4.14) reduces to Yang's equation (3.15).

Since SO(4,1) is a semisimple group we can consider the invariant action

$$I = -\frac{1}{4}\int dV\, K_{ab}\, \tilde{F}^{a}_{\mu\nu}\, \tilde{F}^{b\mu\nu} \qquad (4.15)$$

where K_{ab} is the Killing-Cartan metric. The condition $\delta I = 0$ leads to equations (4.12) and (4.13). In this way we have a gravitational gauge theory for the de Sitter group, which by contraction $(\alpha\to\infty)$, becomes a Poincaré gauge theory.

5. CONCLUSIONS

The results we have obtained may be seen from a different point of view: a gauge theory for the Poincaré group is a de Sitter theory, supplemented by the weak constraints

$$[S,T] \approx 0 \quad , \qquad S \wedge S \approx 0 \;, \qquad (5.1)$$

whose role is to enforce the commutation relations between translations. We could forget about contraction, use the action

$$I = \frac{1}{8}\int \mathrm{tr}\,\left[F \wedge {}^*F - T \wedge {}^*T\right] \qquad (5.2)$$

taking T and S as the "fifth" components in a de Sitter theory (which means that F depends on S for variations) and use (5.1) as weak constraints, which become effective only after the variations are performed.

Further-more, a de Sitter theory **seems** to be quantizable, as an usual gauge theory. This is due to the considerations of weak constraints, imposed at the end of calculations.

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RESUMO

Mostramos uma teoria de gauge gravitacional sem fontes, tendo o grupo de De Sitter como grupo de simetria, considerando um fibrado das bases. As equações gravitacionais de Yang e de Einstein são obtidas a partir de uma teoria clássica de Yang-Mills. O formalismo proposto tem uma integral de ação do tipo gauge, a qual conduz às equações de campo mencionadas.