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Three Non-Degenerate Wave Processes

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We have solved the Heisenberg equations of motion, for the creation operator of three independent optical modes, in a system described by a trilinear Hamiltonian. We did a unitary transformation that is equivalent to the short-time approximation when truncated in the second order. This rendered possible the calculation of the normal ordered characteristic functions of each of the three modes.

Resolvemos a equação de Heisenberg de operadores de criação de três modos ópticos independentes em um sistema descrito por um Hamiltoniano trilinear. Para isto efetuamos uma transformação unitária que quando truncada em 2º ordem gera o mesmo tipo de aproximação que a introduzida pela aproximação do tempo curto ("short-time approximation"). Desta maneira pudemos calcular a função característica ordenada normalmente de cada um dos três modos.

1. GENERAL DESCRIPTION OF THE TRILINEAR MODEL HAMILTONIAN

We consider a general model-Hamiltonian of the Tucker and Wall type[1]. We assume that there are only three interacting modes. The Hamiltonian is written as,

$$H = H_0 + H_I \tag{1}$$

where

$$H_0 = \sum_{i=1}^{3} \hbar\omega_i \; a_i^+ a_i \tag{2}$$

$$H_I = \sum_{ijk} (\hbar K_i\ \xi^{ijk}\ a_i a_j^+ a_k^+ + \hbar K_i^* \xi^{ijk}\ a_i^+ a_j a_k). \qquad (3)$$

ξ^{ijk} is the independent mode selector, that is equal to 1 if $i \neq j \neq k \neq i$ and equal to 0 in all other cases; K_i is a coupling constant and a_i, a_j, a_k are the annihilation operators of the i,j,k modes, respectively, these operators satisfy the commutation relations

$$[a_i, a_j^+] = [a_t(t),\ a_j^+(t)] = \delta_{ij}$$
$$[a_i, a_j] = [a_i^+,\ a_j^+] = 0 \qquad (4)$$

where δ_{ij} is the Kronecker Delta.

In Appendix A a few specific cases are discussed.

2. EQUATIONS OF MOTION

We know that the Heisenberg equation of motion of any operator a which does not depend explicitly on time is given by

$$i\hbar \frac{da(t)}{dt} = [a(t), H] \qquad (5)$$

From the Hamiltonian given in eq. (1) we obtain

$$\frac{da(t)}{dt} = \frac{i}{\hbar} \{[H_0, a(t)] + [H_I, a(t)]\} \qquad (6)$$

Let us introduce the slowly-varying operator $A_0(t)$:

$$a(t) = A_0(t)\ \exp\{-i\omega t/\hbar\}$$

then we can show that

$$\frac{d}{dt}[A_0(t)\ \exp(-i\omega t/\hbar)] = \frac{i}{\hbar}\{[H_0, A_0(t)] + [H_I, A_0(t)]\}\ \exp\{-i\omega t/\hbar\} \qquad (7)$$

and

$$\frac{dA_0(t)}{dt} = \frac{i}{\hbar}\left[H_I, A_0(t)\right] \tag{8}$$

It follows that

$$A_0(t) = \exp\{iH_I t/\hbar\}A_0 \exp\{-iH_I t/\hbar\}. \tag{9}$$

$A_0(t)$ depends on the time in a way that is similar to that given in the interaction representation[3].

Now we use the short-time approximation[1], i.e., we assume that the time of interacting is sufficiently small so that we may expand the generic annihilation operator $A_0(t)$ in a Taylor series and retain terms only up to those quadratic in time:

$$A_0(t) = A_0 + A_0 t + \frac{1}{2}\ddot{A}_0 t \tag{10}$$

Substituting equation (10) in (8) we obtain:

$$(A_0(t) + \ddot{A}_0(t)t) = \frac{i}{\hbar}\left[H_I, A, + \dot{A}_0 t + \frac{1}{2}\ddot{A}_0 t^2\right]$$

$$= \frac{i}{\hbar}\{\left[H_I, A_0\right] + \left[H_I, \dot{A}_0\right]t + \frac{1}{2}\left[H_I, \ddot{A}_0\right]t^2\} \tag{11}$$

In order to find the 1st and 2nd derivatives of $A_0(t)$ we equate the coefficients of the polynomial in t, this yields:

$$\dot{A}_0(t) = \frac{i}{\hbar}\left[H_I, A_0\right] \tag{12}$$

$$A_0(t) = \left(\frac{i}{\hbar}\right)^2\left[H_I, \left[H_I, A_0\right]\right] \tag{13}$$

Let us substitute equations (12) and (13) into eq. (11) to get

$$A_0(t) = A_0 + \frac{i}{\hbar}\left[H_I, A_0\right]t + \frac{1}{2}\left(\frac{it}{\hbar}\right)^2\left[H_I, \left[H_I, A_0\right]\right] \tag{14}$$

that can be put in the form[4]:

$$A_0(t) = \exp\{iH_I t/\hbar\}A_0 \exp\{-iH_I t/\hbar\}. \tag{15}$$

We can see that this approximation is equivalent to (9), within thesort -time approximation.

3. THE TRANSFORMATION METHOD

We introduce a unitary transformation operator

$$S(t) = \exp\{iHt/\hbar\} = \exp\{i(H_0+H_I)/\hbar\} \tag{16}$$

$$S(t)S(t)^{+} = 1 \tag{17}$$

where H is the hamiltonian given by equation (1). By the Baker-Hausdorff identity we can separate $S(t)$ in two parts[5]

$$S(t) = \exp\{iH_0t/\hbar\}\ \exp\{iH_It/\hbar\} \tag{18}$$

since we have $[H_0,H_I] = 0$ (appendix B) due to the conservation of energy condition ($\omega_j = \omega_i+\omega_\ell$). The first part defines the unitary operator that transforms to the interaction representation

$$S_I(t) = \exp\{iH_0t/\hbar\}\ . \tag{19}$$

Then, in conclusion, if we want to find out the time evolution of any operator that obeys the Heisenbera equation of Motion (5), we can use the sirnple forrn

$$a_i(t) = \exp\{iHt/\hbar\}a_i \exp\{-iHt/\hbar\} =$$

$$= \exp\{iH_It/\hbar\}a_i^I(t) \exp\{-iH_It/\hbar\} \tag{20}$$

where $a_i^I(t) = \exp\{iH_0t/\hbar\}a_i \exp\{-iH_0t/\hbar\}$ is the interaction representation of theoperator a_i. Using the identity[4]

$$\exp(xA)B\exp(-xA) = B + x[A,B] + \frac{x^2}{2}[A,[A,B]] + \ldots \tag{21}$$

we find that

$$a_i^I(t) = a_i \exp\{-i\omega_i t\}$$

or

$$a_i(t) = \exp\{iH_I t/\hbar\} a_i \exp\{-iH_I t/\hbar\} \exp\{-i\omega_i t\} =$$

$$= A_i(t) \exp\{-i\omega_i t\}$$

Where, up to the second power in t,

$$A_i(t) = \left\{A_i + \frac{it}{\hbar}\left[H_I, A_i\right] + \frac{1}{2!}\left(\frac{it}{\hbar}\right)^2 \left[H_I, \left[H_I, a_i\right]\right]\right\} \tag{22}$$

then our problem becomes one of doing two commutators.

4. STATISTICAL PROPERTIES

Using this method we can easily find the time evolution of any operator from the general trilinear Hamiltonian (1), that is:

$$a_i(t) = \exp\{-i\omega_i t\} \{a_i + \frac{it}{\hbar}\left[H_I, a_i\right] + \frac{1}{2}!\left(\frac{it}{\hbar}\right)^2 \left[H_I, \left[H_I, a_i\right]\right]\} \tag{23}$$

where

$$\left[H_I, a_i\right] = -\sum_{jk} \xi^{ijk}\, \hbar(2K_j a_j a_k^{+} - K_i^{*} a_j a_k) \tag{24}$$

$$H \quad [H a = -\sum_{i,k,\ell,m} 2\xi^{jk\ell}\{k_j\left[k_\ell \xi^{\ell mi} a_j a_k^{+} a_m^{+} + K_k \xi^{kmi} a_j a_\ell^{+} a_m^{+} -\right.$$

$$- k_m \xi^{mji} a_m a_k^{+} a_\ell^{+} + k_i^{*} \xi^{ikm} (a_j a_\ell^{+} a_m + a_j a_m a_\ell^{+})\Big] +$$

$$+ k_3^{*}\left[k_3 \xi^{jmi} a_m^{+} a_k a_\ell - k_m \xi^{mki} a_j^{+} a_m a_\ell -\right.$$

$$-k_m \xi^{m\ell i} a_j^{+} a_m a_k \quad K_i^{*} \xi^{ijm} a_m a_k a_\ell\Big]\} \tag{25}$$

This is essential in obtaining an expression for the normaly ordered characteristic function, for the j^{th} mode from which we can obtain the statistical properties of the mode[5]. It is defined by

$$C_N(\beta_j) \equiv \mathrm{Tr}\{\rho(0)\exp[\beta_j a_j^+(t)]\ \exp[-\beta_j^* a_j(t)]\} \tag{26}$$

where $\rho(0)$ is the appropriate density matrix[6]. With $a_i(t)$ given in eq. (23), and with the help of eq. (24) and (25), we can see that

$$\exp[-\beta_j^* a_j(t)] = \exp(-\gamma_j^* a_j)\ \exp\left\{-\frac{\gamma_j^* it}{\hbar}\ [H_I, a_j] + \frac{\gamma_j^* t^2}{h}[H_I, [H_I, a_j]]\right\} \tag{27}$$

where $\gamma_j \equiv \beta_j \exp i\omega_j t$. This follows from the Baker-Hausdorff identity and from the easily shown fact that a_i commutes with $[H_I, a_i]$ and with $[H_I, [H_I, a_i]]$, given by equations (24) and (25). Finally, with the short -time approximation in mind, we expand

$$\exp[tA_j + t^2 B_j] \simeq 1 + tA_j + t^2(B_j + \frac{1}{2} A_j^2) \tag{28}$$

where

$$A_j \equiv -\frac{i}{\hbar}\gamma_j^*\ [H_I, a_j]$$
$$B_j \equiv \frac{1}{2\hbar^2}\gamma_j^*\ [H_I, [H_I, a_j]] \tag{29}$$

A similar procedure is followed for $\exp[\beta_j a_j^+(t)]$. One then obtains

$$C_N(\beta_j) = \mathrm{Tr}\{\rho(0)\ \exp(-\beta_j a_j)\ P\ \exp(\beta_j a_j^+)\}, \tag{30}$$

where P is the 2nd degree polynomial in t:

$$P \equiv 1 + t(A_j - A_j^+) + t^2(B_j - B_j^+ - A_j^+ A_j - \frac{1}{2}A_j^2 + \frac{1}{2}A_j^{+2})\ .$$

$C_N(\beta_j)$ is then easily calculated if $\rho(0)$ is written in terms of coherent states.

APPENDIX A

Parametric Amplification[1]

We identify a_1, a_2 and a_3 as the pump mode, the signal mode and the idler mode, respectively a_L, a_S, and a_I).

The Hamiltonian (3) becomes

$$H_0 = \hbar\omega_L a_L^+ a_L + \hbar\omega_S a_S^+ a_S + \hbar\omega_I a_I^+ a_I \qquad (A.1)$$

and

$$H_I = \xi^{LSI}\,\hbar\{K_L(a_L a_S^+ a_I^+) + K_L^* a_L^+ a_S a_I\} + $$
$$+ \xi^{SIL}\,\hbar\{K_S(a_S a_I^+ a_L^+) + K_S^* a_S^+ a_I a_L\} + $$
$$+ \xi^{ILS}\,\hbar\{K_I(a_I a_L^+ a_S^+) + K_I^* a_I^+ a_L a_S\} \qquad (A.2)$$

The coupling constants have to be chosen according to conservation of energy. Putting $K_S = 0$ and $K_I = 0$ we reproduce the Hamiltonian of parametric amplification:

$$H = \hbar\,\omega_L a_L^+ a_L + \hbar\omega_S a_S^+ a_S + \hbar\omega_I a_I^+ a_I +$$
$$+ \hbar\{K_L(a_L a_S^+ a_I^+) + K_L^* a_L^+ a_S a_I\}$$

Various other phenomena, such as generation of Stokes and anti-Stokes[7] and frequency conversion[8], of physical interest, can be described by the same Hamiltonian (1) by suitable choosing the coupling constants and identifying the modes[2].

APPENDIX B

Demonstration that

$$[H_0, H_I] = 0$$

From Eqs. (1), (2) and (3)

$$[H_0, H_I] = \sum_{\substack{i,j\\k,\ell}} \hbar^2 \omega_i K_j \xi^{jk\ell} [a_i^+ a_i ; a_j a_k^+ a_\ell^+] +$$

$$+ \sum_{i,j} \hbar^2 \omega_i K_j^* \ \xi^{jk\ell} [a_i^+ a_i , a_j^+ a_k a_\ell] = \tag{B.1}$$

$$= \sum_{jk\ell} \hbar^2 \ \xi^{jk\ell} \{K_j (\omega_\ell + \omega_k - \omega_j) \ a_j a_k^+ a_\ell^+ +$$

$$+ K_j^* (\omega_j - \omega_\ell - \omega_k) a_j^+ a_k a_\ell\} = \tag{B.2}$$

$$= \sum_{jk\ell} \hbar^2 (\omega_\ell + \omega_k - \omega_j) \xi^{jk\ell} \ \{K_j a_j a_k^+ a_\ell - \text{h.c}\} \tag{B.3}$$

by energy conservation $\omega_j = W_i + \omega_\ell$, then

$$[H_0, H_I] = 0 \tag{B.4}$$

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