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Integral Representations and Addition Theorems for Product of Confluent Hypergeometric Functions Derived by Green's Function

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We show that integral representations and additions theorems for the product of confluent hypergeometric functions can be obtained by means of isotropic harmonic oscillator Green's function.

Obtêm-se representações integrais e teoremas de adição para um produto de funções hipergeométricas confluentes utilizando-se a função de Green para o oscilador harmônico isotrópico multidimensional.

1. INTRODUCTION

In the present paper we show how to use Green's function in order to obtain integral representation and addition theorems for confluent hypergeometric functions. To be able to do that we use the radial Green's function for the N-dimensional isotropic harmonic oscillator.

The N-dimensional isotropic harmonic oscillator Green's function can be put in an integral representation in many ways, in particular by using the generalized Mehler formula for Hermite polynomials[1] or by a quantum mechanical derivation exploiting the $U(N)$ symmetry of the isotropic harmonic oscillator Hamiltonian[2].

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The radial Green's function, in an integral representation, can be derived from a partial wave expansion of total Green's function. A representation in terms of product of two Whittaker functions can be derived from the Sturm-Liouville method[3]. These results are used to derive integral representations and addition theorems for the product of two confluent hypergeometric functions.

2. WHITTAKER'S FUNCTIONS

The Green's function for the N-dimensional isotropic harmonic oscillator satisfies the following differential equation

$$\left[-\frac{\hbar^2}{2\mu}\nabla_N^2+\frac{Kr^2}{2}-E\right]G^N(\vec{r},\vec{r}\,';E)=-\delta(\vec{r}-\vec{r}\,') \tag{1}$$

where ∇_N^2 is the N-dimensional Laplacian operator.

The solution of this differential equation can be obtained through the spectral decomposition in terms of harmonic oscillator wave functions and by using a generalized Mehler formula for the product of Hermite functions[1]

$$\pi^{N/2}\sum_{\nu=0}^{\infty}\xi^{\nu}\sum_{\nu_1+\nu_2+\ldots+\nu_N=\nu}\Psi_{\nu_1}(x_1)\Psi^*_{\nu_1}(x'_1)\ldots\Psi_{\nu_N}(x_N)\Psi_{\nu_N}(x'_N)=$$

$$=(1-\xi^2)^{-N/2}\exp\left\{-\frac{1}{2}(\vec{r}^{\,2}+\vec{r}\,'^2)\right\}\exp\left\{\frac{2\vec{r}\cdot\vec{r}\,'\xi-(\vec{r}^{\,2}+\vec{r}\,'^2)\xi^2}{1-\xi^2}\right\} \tag{2}$$

The solution in a closed integral representation is

$$G^N(\vec{r},\vec{r}\,';\lambda)=\frac{1}{\hbar\omega}\left(\frac{\mu\omega}{\pi\hbar}\right)^{N/2}\exp\left\{\frac{\mu\omega}{2\hbar}(\vec{r}^{\,2}+\vec{r}\,'^2)\right\}\,.$$

$$.\int_0^1 d\xi\,\xi^{-\lambda+N/2-1}(1-\xi^2)^{-N/2}\exp\left\{\frac{\mu\omega}{\hbar}\left[\frac{2\vec{r}\cdot\vec{r}\,'}{1-\xi^2}\xi-\frac{\vec{r}^{\,2}+\vec{r}\,'^2}{1-\xi^2}\xi^2\right]\right\} \tag{3}$$

where $\lambda=E/\hbar\omega$, for $r'>r$ and $\mathrm{Re}(-\lambda+N/2-1)>0$.

We can calculate the radial Green's function if we expand $G^N(\vec{r},\vec{r}\,';\lambda)$ in terms of partial waves. This can be done by expanding

$\exp\{2rr'\xi/(1-\xi^2)-\cos\theta\}$ in a series of Neumann type[4]. The radial Green's function is given by

$$G_L^N(r,r';\lambda) = 2(rr')^{-N/2+1}\exp\left\{-\frac{1}{2}(\vec{r}^2+\vec{r}'^2)\right\}\cdot$$

$$\cdot\int_0^1 d\xi\,\xi^{-\lambda}(1-\xi^2)^{-1}\exp\left\{-\frac{\xi}{1-\xi^2}(\vec{r}^2+\vec{r}'^2)\right\} I_{L+\frac{N}{2}-1}\left(2\sqrt{rr'}\frac{\xi}{1-\xi^2}\right) \qquad (4)$$

with $r' > r$ and $\mathrm{Re}(L+\frac{N}{2}-1) > 0$ where $I_\mu(x)$ is the modified Bessel function. This radial Green's function satisfies the following differential equation

$$\left\{\frac{d^2}{dr^2}+\frac{N-1}{r}\frac{d}{dr}-\frac{L(L+N-2)}{r^2}-r^2+2\lambda\right\}G_L^N(r,r';\lambda) = (rr')^{\frac{1-N}{2}}\delta(r-r') \qquad (5)$$

We can also determine the solution of this differential equation by means of the Sturm-Liouville method in terms of the product of two Whittaker's functions[3]

$$G_L^N(r,r';\lambda) = \Gamma\left(\frac{N}{4}+\frac{L}{2}-\frac{\lambda}{2}\right) M_{\frac{\lambda}{2};\frac{1}{2}(\frac{N}{2}-L+1)}(r)\, W_{\frac{\lambda}{2};\frac{1}{2}(\frac{N}{2}-L+1)}(r') \qquad (6)$$

with $r' > r$ and $\mathrm{Re}\left(\frac{N}{4}+\frac{L}{2}-\frac{\lambda}{2}\right) > 0$.

We can introduce parameters $\mu = \frac{N}{2}+L-1$ and $\nu = \frac{\lambda}{2}$ and make one variable change in eq. (4) $\xi^{-1} = \mathrm{cth}\, v/2$ identifying with eq. (6). We then have:

$$\Gamma(-\nu+\frac{\mu+1}{2})\, M_{\nu;\frac{\mu}{2}}(r)\, W_{\nu;\frac{\mu}{2}}(r') =$$

$$= (rr')^{1/2}\int_0^\infty dv\;\mathrm{cth}^{2\nu}\frac{v}{2}\exp\left\{-\frac{1}{2}(r+r')\,\mathrm{ch}\,v\right\} I_\mu(\sqrt{rr'}\,\mathrm{sh}\,v) \qquad (7)$$

with $r' > r$, $\mathrm{Re}(-\nu+\frac{\mu+1}{2}) > 0$ and $\mathrm{Re}(\mu) > 0$.

As the Whittaker's function are analytical there is no loss of generality by change of parameters. This is the usual integral representation for the product of two Whittaker's functions[5].

We can also determine one addition theorem for the product of two Whittaker's functions by introducing the following Hille-Hardy formula for the modified Bessel functions in eq. (4)

$$(1-\xi^2)^{-1} \exp\left\{\frac{\xi^2+1}{\xi^2-1}\cdot\frac{r+r'}{2}\right\} I_\mu\left(2\sqrt{rr'}\ \frac{\xi}{1-\xi^2}\right) =$$

$$= (rr')^{-1/2} \sum_{n=0}^{\infty} \frac{\Gamma(\mu+n+1)}{\Gamma(n+1)} M_{n+\frac{\mu+1}{2};\mu}(r)\, M_{n+\frac{\mu+1}{2};\mu}(r')\, (\xi^2)^{n+\frac{\mu}{2}} \tag{8}$$

Integrating over the ξ variable, we have the following result:

$$\Gamma(-\nu+\mu+\tfrac{1}{2})\, M_{\nu;\mu}(r) W_{\nu;\mu}(r') =$$

$$= -\sum_{n=0}^{\infty} \frac{1}{\nu-\mu-n-\frac{1}{2}} \frac{\Gamma(2\mu+n+1)}{\Gamma(n+1)} M_{n+\mu+\frac{1}{2};\mu}(r)\, M_{n+\mu+\frac{1}{2};\mu}(r') \tag{9}$$

with $r' > r$, $\mathrm{Re}(-\nu+\mu+\frac{1}{2}) > 0$ and $\mathrm{Re}(\mu) > 0$.

3. PARTICULAR CASES

Functions such as Bessel and Kummer are particular cases of the Whittaker's functions.

The Bessel function is a particular Whittaker function[5] when $\nu=0$

$$M_{0;\mu}(z) = \Gamma(1+\mu)\, 2^{2\mu} z^{1/2} I_\mu(z/2)$$

$$W_{0;\mu}(z) = \pi^{-1/2} z^{1/2}\, K_\mu(z/2) \tag{10}$$

The integral representation for the product of Bessel function $I_\mu(r)$ and $K_\mu(r')$ is

$$I_\mu(r)K_\mu(r') = \int_0^\infty dv \exp\{-(r+r')\text{ch}v\} I_{2\mu}(\sqrt{rr'}\ \text{sh}v) \tag{11}$$

with $r'>r$ and $\text{Re}(\mu+1/2)>0$.

The corresponding addition theorem is:

$$(rr')^{1/2} I_\mu(r/2)\, K_\mu(r'/2) =$$

$$= \sum_{n=0}^{\infty} \frac{1}{n+\mu+1/2} \frac{\Gamma(2\mu+n+1)}{\Gamma(n+1)} M_{n+\mu+1/2;\mu}(r)\ M_{n+\mu+1/2;\ \mu}(r') \tag{12}$$

with $r'>r$, $\text{Re}(\mu+1/2)>0$ and $\text{Re}(\mu)>0$.

Kummer functions are related to Whittaker functions by means of the Kummer transformation[5]

$$\begin{aligned}\Gamma(1+\mu)M_{\nu;\mu/2}(x) &= e^{-x/2}\, x^{\frac{1+\mu}{2}}\ {}_1F_1\left(-\nu+\frac{1+\mu}{2};\ 1+\mu;\ x\right)\\ W_{\nu;\mu/2}(x) &= e^{-x/2}\, x^{\frac{1+\mu}{2}}\ U\left(-\nu+\frac{1+\mu}{2};\ 1+\mu;\ x\right)\end{aligned} \tag{13}$$

Iritroducing the parameters $-\nu=\frac{1+\mu}{2}=a$ and $1+\mu=c$ and by using eq. (7) we have for the Kummer function the following integral representation:

$${}_1F_1(a;c;r)\ U(a;c;r') =$$

$$= \frac{\Gamma(c)}{\Gamma(a)} (rr')^{\frac{1-c}{2}} \int_0^\infty dv\ \text{cth}^{c-2a}\frac{v}{2} \exp\left\{-(r+r')\text{sh}^2\frac{v}{2}\right\} I_{2c-2}(\sqrt{rr'}\ \text{sh}v) \tag{14}$$

where $\text{Re}(a)>0$, $\text{Re}(c-1)>0$ and $r'>r$. We note that these two restrictions on parameters are the usual restrictions to define the Kummer function.

The corresponding addition theorem is

$$\Gamma(\rho)\Gamma(1+2\mu) \quad {}_1F_1(\rho;1+2\mu;r) \quad U(\rho;1+\mu;r') =$$

$$= \sum_{n=0}^{\infty} \frac{1}{n+\rho} \frac{\Gamma(n+1+2\mu)}{\Gamma(n+1)} \quad {}_1F_1(-n;\ 1+2\mu;\ r) \quad {}_1F_1(-n;1+2\mu;r) \tag{15}$$

where we define $\rho = -\nu+\mu+1/2$ and with $r' > r$.

In order to complete the discussion of confluent hypergeometric functions we also present integral representations for Hermite and Laguerre Polynomials.

In this case we only need to calculate the residuum of the isotropic harmonic oscillator Green's function for a defined value of energy.

In the one-dimensional case the residuum of the Green's function is the product of two normalized harmonic oscillator wave functions

$$\operatorname*{Res}_{\lambda=n+1/2} G(x,x';\lambda) = \Psi_n(x) \ \Psi_n^*(x') \tag{16}$$

A single residuum calculation of eq.(3) shows that

$$\operatorname*{Res}_{\lambda=n+1/2} G(x,x';A) = \pi^{-1/2} \exp\left\{-\frac{1}{2}(x^2+x'^2)\right\}.$$

$$\cdot \frac{1}{2\pi i} \oint_{\xi=0} d\xi \ \xi^{-(n+1)} (1-\xi^2)^{-1/2} \exp\left\{\frac{2xx'}{1-\xi^2}\xi - \frac{x^2+x'^2}{1-\xi^2}\xi^2\right\} \tag{17}$$

with $x' > x$ and $\mathrm{Re}(n+1) > 0$. The integral representation for the product of two Hermite polynomials is

$$\frac{2^{-n}}{\Gamma(n+1} H_n(x) \ H_n(x') = \frac{1}{2\pi i} \oint_{\xi=0} d\xi \ \xi^{-(n+1)} (1-\xi^2)^{-1/2} \exp\left\{\frac{2xx'}{1-\xi^2}\xi - \right.$$

$$\left. - \frac{x^2+x'^2}{1-\xi^2}\xi^2\right\} \tag{18}$$

where $x' > x$ and $\mathrm{Re}(n+1) > 0$.

For the Lagrange polynomials we can calculate the residuum of the b'i-dimensional Green's function as follows:

$$\underset{\lambda=n+1}{\text{Res}} \quad , \vec{\ } \vec{\ } ;\lambda) = \frac{1}{\pi} \exp\left\{-\frac{1}{2}(\vec{r}^2+\vec{r}'^2)\right\}$$

$$\cdot \frac{1}{2\pi i} \oint_{\xi=0} d\xi\, \xi^{-(n+3/2)}(1-\xi^2)^{-1} \exp\left\{\frac{2\vec{r}.\vec{r}'}{1-\xi^2}\xi - \frac{\vec{r}^2+\vec{r}'^2}{1-\xi^2}\xi^2\right\} \tag{19}$$

with $r'>r$ and $\text{Re}(n+3/2) > 0$

In this case the residuum of the Green's function is given by

$$\underset{\lambda=n+1}{\text{Res}} \quad G(\vec{r},\vec{r}';\lambda) \quad \sum_{2k+|m|=n}\sum \quad \Psi_{k,m}(\vec{r})\Psi^*_{k,m}(\vec{r}') \tag{20}$$

where k is the radial quantum number and m is the azimuthal quantum number. Using the wave function in polar coordinates[6] we have

$$\underset{\lambda=n+1}{\text{Res}} \quad G(\vec{r},\vec{r}';\lambda) =$$

$$= \sum_{2k+|m|=n}\sum \frac{\Gamma(k+1)}{\Gamma(k+|m|+1)} (rr')^{|m|} L_k^{|m|}(r^2) L_k^{|m|}(r'^2)\, e^{im(\phi-\phi')} \tag{21}$$

If we expand exp. $\{2rr' \frac{\xi}{1-52} \cos(\phi-\phi')\}$ in a Bessel series[4] and use the constraint $k = \frac{n}{2} - |m|$ we get,

$$\frac{\Gamma(\frac{n}{2}-m+1)}{\Gamma(\frac{n}{2}-1)} (rr')^m L^m_{\frac{n}{2}-m}(r^2)\, L^m_{\frac{n}{2}-m}(r'^2) =$$

$$= \frac{1}{2\pi i}\oint_{\xi=0} d\xi\, \xi^{-(n+3/2)} (1-\xi^2)^{-1} I_m \frac{2rr'\xi}{1-\xi^2} \exp\left\{-\frac{\vec{r}+\vec{r}'^2}{1-\xi^2}\xi^2\right\} \tag{22}$$

with $r' > r$, $\text{Re}(n/2 - m) > 0$ and $\text{Re}(n/2 - 1) > 0$.

The integral representation in eq. (18) can also be derived directly from the generalized Mehler formula.

We note that the generalized Mehler formula can be used to calculate the time-dependent isotropic harmonic oscillator Green's function, if we change $\xi = \exp(it)$ and multiply both sides of eq. (3) by $\exp(i\frac{N}{2})$, thus getting

$$\pi^{-N/2} \sum_{\nu=0}^{\infty} e^{-i(\nu+\frac{N}{2})t} \sum_{\nu_1+\nu_2+\ldots\nu_N=\nu} \Psi_{\nu_1}(x_1)\Psi^*_{\nu_1}(x'_1)\ldots\Psi_{\nu_N}(x_N)\Psi^*_{\nu_N}(x'_N) =$$

$$= (2i\ \sin t)^{-N/2} \exp\left\{-\frac{i}{\sin t}\left[\vec{r}_0\vec{r}\,' - (\vec{r}^{\,2}+\vec{r}\,'^{2})\cos t\right]\right\} \qquad (23)$$

which is the time-dependent isotropic harmonic oscillator Green's function.

REFERENCES

1. J.Bellandi Filho and E.S.Caetano Neto, J.Phys.A Math.Gen. *9*, 683 (1976).
2. G.Berendt and E.Weimar, Lett. Nuovo Cimento 5, 613 (1972).
3. E.Capelas de Oliveira, Rev.Bras.Fis. *10*, 259 (1980).
4. Bateman Manuscript Projects, *Higher Transcedental Function-Edited* by A.Erdélyi (vol.1-2), New York, NY (1953).
5. H. Buchholtz, The *Confluent Hypergeometric Function* (Springer-Verlag Berlin, 1969).
6. J.L.Powell and B.Crasemann, *Quantum Mechanics* (New York, Addison--Wesley-1965).