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Para-Bosons and Para-Fermions in Quantum Mechanics

M. CATTANI and N. C. FERNANDES

Instituto de Física, Universidade de São Paulo, Caixa Postal 20516, SP, Brasil

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We have performed, within the framework of the ordinary quantum mechanics, a detailed study of the energy eigenfunctions of N identical particles using the irreducible representations of the permutation group in the Hilbert space. It is shown that the para-states, as occurs with the boson and fermion states, are compatible with the postulates of quantum mechanics and with the principle of indistinguishability. This paper gives a mathematical support for the existence of para-bosons and para-fermions and justifies, in a certain sense, Gentile's quantum statistics.

Fazemos um estudo pormenorizado, dentro do formalismo da mecânica quântica ordinária, das auto-funções da energia de N partículas idênticas, usando as representações irredutiveis do grupo de permutações no espaço de Hilbert. Mostramos que os para-estados, assim como ocorre com os bósons e os férmions, são compatíveis com os postulados da mecânica quântica e com o princípio da indistinguibilidade. Este trabalho fornece um suporte matemático para a existência de para-bósons e para-férmions justificando, de um certo modo, a estatística quântica de Gentile.

1. INTRODUCTION

About four decades ago Gentile[1-3] invented, without any quantum-mechanical or another type of justification, a parastatistics. He has obtained a statistical distribution function for a system of N "weakly interacting" particles assuming that the quantum states of an individual particle can be occupied by an arbitrary finite number d of particles. The Fermi and the Bose statistics are particular cases of this parastatistics for $\underline{d} = 1$ and $\underline{d} = \infty$, respectively.

Ten years later, Green[4] has shown that in quantum field theory, the para-Bose and the para-Fermi quantizations, considered as generalizations of the Bose and Fermi quantization, were theoretically possible. After this, many papers[5-10] have been written about the para-particles in the domain of the quantum field theory. Messiah and Greenberg[11] have also analysed this problem from the usual quantum-mechanical standpoint.

In this work we perform a detailed study, in the ordinary quantum-mechanical approach, of the energy eigenfunctions of N identical particles using the irreducible representations of the permutation group in the Hilbert space. Analysing these energy eigenfunctions we arrived at the following conclusions:

1) The para-states, as occurs with the boson and the fermion states, are compatible with the postulates of the Quantum Mechanics and with the Principle of Indistinguishability.

2) In the limit of weakly interacting para-boson or para-fermions, the occupation number $\underline{d}$, for the quantum states of individual particles, can be an arbitrary finite number.

This analysis, which gives support to the mathematical existence of para-fermions and para-bosons within the framework of quantum mechanics, justifies, in a certain sense, Gentile's statistics.

2. THE IRREDUCIBLE REPRESENTATIONS OF THE PERMUTATION GROUP IN THE HILBERT SPACE

Let us consider an isolated system, with total energy, E, composed by a constant number N of identical particles that is described by the particle quantum mechanics.

If H is the Harniltonian operator of the system, the energy eigenfunction ψ, that obeys the equation $H\psi = E\psi$, is given $\psi = \psi(\vec{x}_1, s_1, \ldots, \vec{x}_N, s_N)$, where $\vec{x}_i$ and s_i denote the position coordinate and the spin orientation, respectively, of the i^{th} particle. We abbreviate the pair $(\vec{x}_i, s_i)$ by a single number $\underline{i}$ and call $1,2,\ldots,N$ a particle configuration. The set of all possible configurations will be called the configuration space $\varepsilon^{(N)}$.

Tlie quantum states ψ of the system composed by N identical particles are described by tensors[12] in a Hilbert space $L_2(\varepsilon^{(N)})$ of all square integrable functions over $\varepsilon^{(N)}$.

The permutations P_i ($i = 1,2,\ldots,N!$) of the labels $1,2,\ldots$, N of $\varepsilon^{(N)}$ constitute the symmetric group $S^{(N)}$ of order N! To each P_i we can associate, in a one-to-one correspondence, an unitary operator $U(P_i)$ in the $L_2(\varepsilon^{(N)})$.

Starting from the general substitutional expression for the permutations P_i [13].

$$\pi = a_1 P_1 + a_2 P_2 + \ldots + a_{N!} P_{N!} \tag{1}$$

where $P_1 = E$ is the identity permutation and the a_i are numercial coefficients, we construct the unitary operator $U(\pi)$ such as

$$U(\pi)\psi = \psi_\pi = a_1 \psi_1 + \ldots + a_{N!} \psi_{N!} \tag{2}$$

Now, it is well known from group theory that we can construct N! linearly independent substitutional expressions of the form (1) and that any substitutional expression can be written in terms of them.

We associate to the linear operators $P_i, P_j, \ldots$, in a one-to-one correspondence, a set of matrices $H(P_i)$, $H(P_j), \ldots$, such that $H(P_i).H(P_j) = H(P_iP_j)$. This set is a representation of $S^{(N)}$ in $\varepsilon^{(N)}$.

The representations of $S^{(N)}$ in $\varepsilon^{(N)}$ are related to the partitions of the number N. Any partition of N will be denoted by $[\alpha_1, \alpha_2, \ldots, \alpha_k]$, where $\alpha_1 + \alpha_2 + \ldots + \alpha_k = N$, with $\alpha_1 \geqslant \alpha_2 \geqslant \ldots \geqslant \alpha_k$. In what follows, when no confusion is likely to arise, we denote the partition simply by (α). Of course, an irreducible representation of

$S^{(N)}$ corresponding to a partition (α) of N is not unique, although all such representations are equivalent. In this work we shall use the natural representation.

Now we specify the $N!$ linearly independent substituional expressions π that will constitute the basic set of units, in terms of which any arbitrary substitutional expression can be decomposed.

The natural units, in an irreducible representation (α), of the symmetric group $S^{(N)}$ are given by[13]

$$g_{rs}^{(\alpha)} = (1/\theta^{(\alpha)}) . \sum_t E_{rt}^{(\alpha)} \eta_{ts}^{(\alpha)} \tag{3}$$

where $\theta^{(\alpha)} = N! / f^{(\alpha)}$, $f^{(\alpha)}$ the dimension of the square sub-matrix associated to a given partition (α), $E_{rt}^{(\alpha)}$ $(r,t = 1,2,...,f^{(\alpha)})$ are convenient substitutional expressions to be defined in the sequence and $\eta_{ts}^{(\alpha)}$ are the matrix elements of H. The $f^{(\alpha)}$ satisfy the Frobenius theorem $\sum_a (f^{(\alpha)}) = N!$.

When $N \leq 4$, the calculations are simplified and the equation (3) can be read:

$$g_{rs}^{(\alpha)} = (1/\theta^{(\alpha)}) \; E_{rs}^{(\alpha)} \tag{4}$$

Corresponding to any partition (α) of the number N, a certain arrangement of N spaces, called a shape, can be constructed having α_1 spaces in the first row, α_2 in the second an so on. By permuting the N positions we get N! different arrangernents. Each spatial arrangement is called a tableau of the given shape. Of the $N!$ tableaux of the shape (α) there will be a certain number $f^{(\alpha)}$ which have the property thet the numbers in each row in each column are in crescent order. Such tableaux are called standard tableaux. If, for a given partition (α), we define P_r as an element of the positive symmetric group of the rows of a standard tableau and $N_t^{(\alpha)}$ as an element of the negative symmetric group of the colurnns, the $E_{rt}^{(\alpha)}$ are defined by the product $E_{rt}^{(\alpha)} = P_r^{(\alpha)} N_t$. This completes the definition of $E_{rt}^{(\alpha)}$. To obtain the natural units $g_{rs}^{(\alpha)}$, given by equation (3), it is enough to calculate the expressions $E_{rt}^{(\alpha)}$, which is a straightforward procedure.

The one-to-one correspondence between π and $U(\pi)$, defined by equations (1) and (2), respectively, implies a similar correspondence between the units $g_{rs}^{(\alpha)}$ and the wavefunctions $\Psi_{rs}^{(\alpha)}$, that, in the natural representation are given by

$$\Psi_{rs}^{(\alpha)} = C^{(\alpha)} \left[\sum_{j=1}^{N!} \xi_j^{(\alpha)}(r,s)\, \Psi_j \right] \tag{5}$$

where $C^{(\alpha)}$ is a normalization constant, $\xi_j^{(\alpha)}(r,s)$ are coefficients that assume the values 0, +1 and -1, and the wavefunctions $[\Psi_1, \Psi_2, \ldots, \Psi_{N!}]$ are base-vectors in $L_2(\varepsilon^{(N)})$.

The tensors $\Psi^{(\alpha)} \equiv \left[\Psi_{rs}^{(\alpha)}\right]$, with $r, s = 1,2,\ldots,f^{(\alpha)}$, are the irreducible representations (α) of $S^{(N)}$ in $L_2(\varepsilon^{(N)})$. The wavefunctions $\Psi_{rs}^{(\alpha)}$, that obey the equation $H\Psi_{rs}^{(\alpha)} = E\Psi_{rs}^{(\alpha)}$, belong to an irreducible sub-space $h^{(\alpha)}$ of $L_2(\varepsilon^{(N)})$. The dimension of $h^{(\alpha)}$ is $(f(\alpha))^2$.

Since the sub-spaces $h^{(\alpha)}$ and $h^{(\beta)}$, with $\alpha \neq \beta$ are irreducible, the scalar product $\langle\Psi_{rs}^{(\alpha)} | \Psi_{tv}^{(\beta)}\rangle = 0$.

There are two irreducible sub-spaces of special interest: (a) = $= (N)$ and $(\alpha) = (1^N)$. Since in both cases $f^{(\alpha)} = 1$, the corresponding sub-spaces are one-dimensional. The wavefunctions associated to them are, respectively:

$$\Psi_{11}^{(N)} = (1/N!)^{1/2} \sum_{j=1}^{N!} \Psi_j = \Psi_S \tag{6}$$

$$\Psi_{11}^{(1^N)} = (1/N!)^{1/2} \sum_{j=1}^{N!} \delta_{P_j} \Psi_j = \Psi_A \tag{7}$$

where $\delta_{P_j} = \pm 1$, if P_j is an even or an odd permutation, respectively.

There are only two one-dimensional sub-spaces. In the remaining sub-spaces h $^{(\alpha)}$, with dimensions going from 2^2 up to $(N-1)^2$, the functions $\Psi_{rs}^{(\alpha)}$ are symmetric with respect to some permutations, anty-symmetric with respect to others and have an indefinite symmetry with respect to the remaining ones.

To illustrate the above results, we consider the simplest non trivial case of N=3. In this case the dimension of $L_2(\varepsilon^{(3)})$ is $N! = 6$. Indicating by $[\Psi_1, \Psi_2, \ldots, \Psi_6] = u(123), u(132), u(213), u(231), u(312), u(321)$ the base vectors of $L_2(\varepsilon^{(3)})$, the wavefunction $\Psi_{11}^{(3)} = \Psi_S$ that belongs to the sub-space $h_2^{(3)}$, associated to the tableau $\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline\end{array}$ is given by:

$$\Psi_S = (1/6)^{1/2} \; (u(123) + u(132) + \ldots + u(321)) \tag{8}$$

The totally anti-symmetric function $\Psi_A = \Psi_{11}^{(1^3)}$, associated to the tableau $\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline\end{array}$ is written as

$$\Psi_A = (1/6)^{1/2} \; (u(123) - u(132) - u(213) + u(231) + u(312) - u(321)) \tag{9}$$

Besides these two one-dimensional sub-states there is only one sub-space, with dimension 4, which is associated to the tableaux $\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}$ and $\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}$. The wavefunctions corresponding to this-space are:

$$\begin{aligned}
\Psi_{11} &= (1/2)(u(123) + u(213) - u(231) - u(321)) \\
\Psi_{12} &= (1/2)(u(132) - u(213) + u(231) - u(312)) \\
\Psi_{21} &= (1/2)(u(132) - u(231) + u(312) - u(321)) \\
\Psi_{22} &= (1/2)(u(123) - u(213) - u(312) + u(321))
\end{aligned} \tag{10}$$

These functions, as one can easily verify, obey the properties of symmetry cited above.

3. BASE VECTORS OF THE IRREDUCIBLE SUB-SPACES $h^{(\alpha)}$

Since the $\Psi_{rs}^{(\alpha)}$ $(r,s=1,2,\ldots,f^{(\alpha)})$ forrn a set of linearly independent functions in $h^{(\alpha)}$ we can construct by an orthonormalization process, the base-vectors of the sub-space $h^{(\alpha)}$ which will be denoted by the column vector $Y(\alpha)$:

$$Y(\alpha) = (1/\tau)^{1/2} \begin{pmatrix} Y_1(\alpha) \\ Y_2(\alpha) \\ \ldots\ldots \\ Y_\tau(\alpha) \end{pmatrix} \qquad (11)$$

where $\tau = (f^{(\alpha)})^2$ is the dimension of $h^{(\alpha)}$.

The base vectors of the one-dimensional sub-spaces are given simply by $Y_A = \Psi_A$ and $Y_S = \Psi_S$, where Ψ_A and Ψ_S are defined by equations (6) and (7).

We will show now that all physical properties of our N-particle system represented by a given sub-space $h^{(\alpha)}$ can be obtained by using the base-véictors $Y(\alpha)$ satisfying both the probabilistic interpretation of quantum mechanics and the principle of indistinguishability.

A given perrnutation P of the particle in $\varepsilon^{(N)}$ is represented by an unitary operator $U(P)$ in $L_2(\varepsilon^{(N)})$. Thus, under the permutations, the base-vectors $Y(\alpha) \in h^{(\alpha)}$ is changed into a vector $X(\alpha) \in h^{(\alpha)}$ given by $X(\alpha) = U(P)Y(\alpha)$. This permutation operation can also be represented by an unitary matrix $T_\alpha : X(\alpha) = T_\alpha\, Y(\alpha)$. Since the irreducible sub-spaces are equivalence classes[14] different sub-spaces have different symmetry properties which are defined by T_α matrix. This means that if $T_\alpha \in h^{(\alpha)}$ and $T_\beta \in h^{(\beta)}$, results $T_\alpha \neq T_\beta$ if $\alpha \neq \beta$.

For the one-dimensional cases, since $X_A = U(P)\ Y_A = -Y_A$ and $X_S = U(P)\ Y_S = Y_S$, the T rnatrices have only one component $T_A = -1$ and $T_S = +1$. The permutation of particles changes the state-vectors only by a numerical factor; X and Y belong to the same ray in the corresponding sub-spaces. We see that the permuted state-vector X and the original one Y describe the same physical state, that is $|X|^2 = |Y|^2$. This permit us to interpret the permutation invariant function $|Y|^2$ as the probability density function.

For a multi-dimensional $h^{(\alpha)}$, since $T^{+}_{\alpha}T_{\alpha} = 1$, the square modulus of $Y(\alpha)$ is permutation-invariant, that is, $Y^{+}(\alpha)\, Y(\alpha) = X^{+}(\alpha)X(\alpha)$. So, for these cases, the function $|\phi^{(\alpha)}|^2 = Y^{+}(\alpha)Y(\alpha) = \sum_{i=1}^{\tau} |Y_i(\alpha)|^2$ can be interpreted as the probability density function.

We note that for the one-dimensional cases the symmetry properties of Y are very simple because $T = \pm 1$, whereas for the multi-dimensional $h^{(\alpha)}$, the symmetry properies are not so evident because they are defined by T_{α} which has $[f^{(\alpha)}]^2$ components.

To obtain the energy eigenfunctions ou basic hypothesis was that $[U(P),H] = 0$. Consequently, $[U(P), S(t)] = 0$, where $S(t)$ is the time evolution operator for the system.

The expectation values of an arbitrary Hermitean operator $A = A(1,2,\ldots,N)$ for the energy state-vectors $Y(\alpha)$ and $X(\alpha)$ are defined by $\bar{A}_Y = \langle Y(\alpha)|A|Y(\alpha)\rangle = (1/\tau)\sum_{i=1}^{\tau} \langle Y_i(\alpha)|A|Y_i(\alpha)\rangle$ and $\bar{A}_X = \langle X(\alpha)|A|X(\alpha)\rangle = (1/\tau)\sum_{j=1}^{\tau} \langle X_j(\alpha)|A|X_j(\alpha)\rangle$, respectively. Since $X(\alpha) = T\,Y(\alpha)$, we see that $\bar{A}_X = \langle X(\alpha)|A|X(\alpha)\rangle = \langle Y(\alpha)|T^{+}_{\alpha} A\, T_{\alpha}|Y(\alpha)\rangle = \langle Y(\alpha)|A|Y(\alpha)\rangle = \bar{A}_Y$, implying that $[U(P),A] = 0$. Moreover, if $U(P)$ commutes with $S(t)$, the relation $[U(P),A(t)] = [U(P), S^{+}(t)AS(t)] = 0$ is satisfied. This means that $\bar{A}_Y(t) = \bar{A}_X(t)$ at any instant of the time. This expresses the fact that since the particles are identical, any permutations of them does not lead to any observable effect. This conclusion is in agreement with the postulate of indistinguishability[11].

In the limit of weakly interacting particles let is indicate by $\alpha,\beta,\gamma,\ldots$ the individual states that can be assumed by the particles. So, writing the base vectors $|\Psi_i|$ $(i=1,2,\ldots,N!)$ of $L_2(\varepsilon^{(N)})$ in the form $\Psi_i(m,n,p,\ldots) = u_{\alpha}(m)u_{\beta}(n)u_{\gamma}(p)\ldots$, one can determine the maximum value $\underline{d}$ for the occupation number of a given state. We verify that for Y_S, $d = N$, which can be arbitrarily large, $d = 1$ for Y_A and d goes from 2 up to $N-1$ for the state-vectors $Y(\alpha)$ of the multi-dimensional subspaces $h^{(\alpha)}$. That is, for a system of weakly interacting particles, the maximum value $\underline{d}$ for the occupation number of an individual quantum state can be $d = 1,2,3,\ldots,N$. It is equal to the number of spaces α_1 of the first row of the Young shape that is associated to $h^{(\alpha)}$.

We must note that the general solution of the equation HY= EY, cornpatible with the principles of quanturn rnechanics and the the postulate of indistinguishability of the particles, should be given by a linear superposition of the state-vectors $Y(\alpha)$: $\Psi = \sum_{\alpha} C_{\alpha} Y(\alpha)$, where are arbitrary numerical constants. This general eigenfunction should be a column-vector with $N!$ rows composed by the colurnn-vectors $Y(\alpha)$, each one with $(f^{(\alpha)})^2$ rows.

To illustrate this section we return to study the case of $N=3$. To sirnplify the notation we write the set of linearly independent functions Ψ_{rs}, defined by the equation (10), as a "colurnn-vector" $[\chi]$:

$$[\chi] = \begin{pmatrix} \Psi_{11} \\ \Psi_{12} \\ \Psi_{21} \\ \Psi_{22} \end{pmatrix} \tag{12}$$

The base-vector Y of the 4-dimensional sub-space, constructed orthogonalizing $[\chi]$ is given by the linear combination of the Ψ_{rs}:

$$Y = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1/\sqrt{3} & 2/\sqrt{3} & 0 & 0 \\ -1/\sqrt{3} & 0 & 2/\sqrt{3} & 0 \\ -1/\sqrt{3} & -2/3 & 2/3 & 4/3 \end{pmatrix} \begin{pmatrix} \Psi_{11} \\ \Psi_{12} \\ \Psi_{21} \\ \Psi_{22} \end{pmatrix} . \qquad S[\chi]$$

Applying an arbitrary perrnutation P, the set $[\chi]$ is changed into a new set $[\chi']$:

$$|\chi'| = \begin{pmatrix} \alpha & \beta & 0 & 0 \\ \gamma & \delta & 0 & 0 \\ 0 & 0 & \alpha & \beta \\ o & o & y & o \end{pmatrix} \begin{pmatrix} \Psi_{11} \\ \Psi_{12} \\ \Psi_{21} \\ \Psi_{22} \end{pmatrix} = P|\chi|$$

Indicating by P_i $(i=1,2,\ldots,6)$ the corresponding permutations, we get

$\alpha = 1$, $\beta = 0$, $\gamma = 0$ and $\delta = 1$ for P_1
$\alpha = 0$, $\beta = 1$, $\gamma = -1$ and $\delta = -1$ for P_2
$\alpha = -1$, $\beta = -1$, $\gamma = 1$ and $\delta = 0$ for P_3
$\alpha = 0$, $\beta = 1$, $\gamma = 1$ and $\delta = 0$ for P_4
$\alpha = -1$, $\beta = -1$, $\gamma = 0$ and $\delta = 1$ for P_5
and $\alpha = 1$, $\beta = 0$, $\gamma = -1$ and $\delta\ 1$ for P_6

With this new set $[\chi']$ of linearly independent functions we obtain the base-vectors $X = S[\chi']$. Thus, using the above relations, finally we obtain the relation $X = TY$ as follows:

$$X = S[\chi'] = SP[\chi] = SPS^{-1}\, Y \equiv TY \ ,$$

where $T = SPS^{-1}$ is unitary, as can be easily verified.

In the limit of weakly interacting particles, $\underline{d} = 3$ for $\Psi = Y_S$, $\underline{d} = 2$ for $\Psi = Y$ and $\underline{d} = 1$ for $\Psi = Y_A$.

4. CONCLUSIONS

Due to the indistinguishability of the N particles, according to sections 2) and 3), the energy eigenvalue E is $N!$ degenerate. The energy eigenfunctions $Y(\alpha)$ that belong to the irreducibles sub- spaces $h^{(\alpha)}$ are column-vectors with τ rows where $\tau = (f^{(\alpha)})^2$ is the dimension of $h^{(\alpha)}$. The general solution of the equation $H\Psi = E\Psi$ is a column- vector Ψ with $N!$ rows, given by the linear superposition of the state-vectors $Y(\alpha)$:

$$\Psi = \sum_\alpha C_\alpha\, Y(\alpha) = C_S Y_S + C_A Y_A + \sum_\alpha{}' \ C'_\alpha \ Y'(\alpha) \ ,$$

where we have put in evidence the one-dimensional representations Y_A and Y_S and we have denoted the multi-dimensional representations by $Y'(\alpha)$.

We must note that it is not possible to determine, by any quantum-mechanical consideration, the values that can be assumed by the constants C_α.

iiowever, since the sub-spaces $h^{(\alpha)}$ are equivalence classes it seems reasonable to expect that the particles with a common characteristic should be represented by a specific $Y(\alpha)$. Indeed, up to now, for all particles that have been tested experimentally ψ is only given by $\psi = Y_S$ or by $\psi = Y_A$. In the first case, the particles, called bosons, have in common an integral spin. In the second case, the fermions, an odd half-integral spin.

Thus, the bosons are represented by Y_S and the fermions by Y_A. The state-vectors $Y'(\alpha)$ of the multi-dimensional sub-spaces should represent particles that are neither bosons nor fermions. They are called para-bosons if they have integral spin and para-fermions if they have an odd half-Integral spin[4,7,8].

Since only bosons and fermions have been observtd, many papers[15-18] have been written to prove that only totally syrnmetricor only totally anti-symmetric functions can exist in quanturn mechanics if the indlstinguishability of the particles is assumed. However, the arguments that have been adopted for the proof are not cornpletely satisfactories since, as it was shown by Messiah and Greenberg[11] and by Haag[19], they are equivalent to impose the one-dimensionality of the eigenfunctions Y.

Of course, it is quite possible that all physical particles obey the ordinary quantum statistics and that para-bosons and para--fermions do not exist in nature. This should be an enormous simplification of the N-identical particles problem because all multi-dimensional representations are eliminated, remaining only two very simple one-dimensional representations.

As, at least in principle, there are no theoretical inconsistencies at the level of wave mechanics in adopting any irreducible representation sub-space for describing the quantum states, we are obliged to accept an "a priori" argument to rule out the inconvenient representations. We adopt in this case the Symmetrization Postulate which can be interpreted as a supplementary condition for the quantum problem.

The Gentile parastatistics[1-3] was developed to treat particles which have an arbitrary finite occupation number d. The fermion and boson descriptions are obtained as particular cases of this parastatistics for d=l and $d=\infty$, respectively. The para-particles should be described for a finite $d>1$.

Even assuming that the para-particles do not exist in nature, in our opinion, Gentile's approach is very important as an improvement of Bose statistics. It is able to describe more realistically and more accurately the systems composed of a finite number N of bosons[1-3,20,21]. In this sense, Bose statistics should be rigorously valid only in the limit of $N\to\infty$ as it occurs, for instance, with the photons in a cavity[22].

In quantum field theory, specific theoretical models[4,5,6-8] have been proposed to see if all particles obey either Bose or Fermi statistics. Among the alternatives for the problem[23] we quote that Green[4] showed that quantum statistics can be considerably generalized if one quantizes fields according to a system of axioms that abandon the usually accepted c-numbers postulate, i.e., the requirement for the commutador or the anti-commutator of two field to be a c-number. A strong indication in support of Green's parastatistics conjecture is given by the decomposition of a parafield. Thus, for instance, a para-Fermi field of order p may be written as the sum of p mutually commuting Fermi fields. The observables should be functions of the parafield and the theoretical possibilities for their selection are restricted by the principle of locality. By adopting the point of view of Doplicher, Haag and Roberts[24,25], if the net U of algebras of local observables is the basic mathematical object of the theory, and if we consider a set C as states over U as representing the states of interest in elementary particle physics, it is possible to show that the pure states of this set are subdivided into superselection sectors. Each superselection sector is labelled by generalized charge quantum numbers and possesses a "statistics parameters" λ which determines the nature of the representations of the group $S^{(n)}$ of the permutations on n elements, for all n. This group is analogous to that considered in sections 2) and 3), which arises in wave mechanics when permuting the arguments of the N-particles state-vectors.

Thus, if taken in a very cautions sense, we can follow the analogy, and say that Gentile has antecipated the formulation of parastatistics in the scheme of wave mechanics. It is also interesting to note that the assurnption of a finite d in the quantum statistics of Gentile is compatible with the occupation numbers deduced by Green from his quantum theoretical field reasoning. However, to pursue such an analogy is outside the scope of the present work and only this brief remark is permited here.

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