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On Cartan's Equations on Complex Manifolds

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We discuss Cartan's structure equations on complex manifolds and deduce holomorfic expressions for curvature defined over this manifold. It's possible to construct complex versions of Einstein field equations by contractions of this curvature.

Discutimos as equações de estrutura de Cartan em uma variedade complexa e deduzimos expressões holomorfas para a curvatura definida sobre essa variedade. É possível se construir versões complexas das equações de campo de Einstein através de contrações da curvatura.

1. INTRODUCTION

It's well known[1] that complex versions of Einstein's equations are easier to solve than their real forms. We consider here a complex manifold defined with complex coordinates $z^a = x^\alpha + ix^a$ where runs from 1 to n and $\bar{\alpha}$ from $n+1$ to $2n$ (n is a positive integer). Over this manifold define a metric structure[2]

$$ds^2 = g_{ab}\, dz^a\, dz^b = 2g_{\alpha\bar{\beta}}\, dz^\alpha \otimes \overline{dz^\beta}$$

where bar over z denotes complex conjugation and $\otimes$ is the usual tensorial product. In this way $g = 2g_{\alpha\bar{\beta}}\, dz^a \otimes dz^{\bar{\beta}}$ defines a covariant metric of (1,1) type. We use two different kinds of exterior derivative besides that operator d used in real riemannian manifolds. They are:

$$\partial \equiv dz^{\alpha} \frac{\partial}{\partial z^{\alpha}}$$

$$\bar\partial \equiv \overline{dz^{\beta}} \frac{\partial}{\partial \bar z^{\beta}}$$

Flaherty has shown that $d = \partial + \bar\partial$. [3]

Over this manifold we use differeiitial forms of types $(0,1)$ and $(1,0)$:

i) type $(1,0)$: $\omega = \Omega_a \, dz^a$

ii) type $(0,1)$: $\omega = \Omega_{\bar a} \, \overline{dz^a}$

forms of higher degree are obtained through the exterior product So we can form various types of 2-forms:

$$\omega = \omega_{\alpha\bar\beta} \, dz^{\alpha} \wedge \overline{dz^{\beta}} \quad , \qquad \omega = \omega_{\alpha\beta} \, dz^{\alpha} \wedge dz^{\beta}$$

$$\omega = \omega_{\bar\alpha\beta} \, \overline{dz^{\alpha}} \wedge dz^{\beta} \quad , \text{ or } \qquad \omega = \omega_{\overline{\alpha\beta}} \, \overline{dz^{\alpha}} \wedge \overline{dz^{\beta}}$$

mathematical structures like $g_{\alpha\bar\beta}$, $\omega_{\overline{\alpha\beta}}$, $\omega_{\alpha\bar\beta}$, $\omega_{\bar\alpha\beta}$ are all holomorphic functions of coordinates z^a and $z^{\bar a}$.

2. CARTAN'S EQUATIONS ON COMPLEX FIIEMANNIAN MANIFOLDS

Starting with an n-dimensional real analytic manifold N one construct an n-dimensional complex manifold by allowing the coordinates to take complex values and analytic extending the coordinate transformation maps. The resulting complex manifold M is also a $2n$-dimensional real analytical manifold in which N sits as an n-dimensional submanifold, given locally by:

$$z^{\alpha} = \overline{z^{\alpha}} \quad \text{or} \quad x^{\bar\alpha} = 0 \qquad (2.1)$$

in the analytically continued coordinates $z^a = x^a + ix^{\bar a}$, x^a and $x^{\bar a}$ are real coordinates. (For a discussion of real slices on complex space--time see ref. (2)). We dotted this complex manifold with a 1-form conexion:

$$\Gamma^{\alpha}{}_{\beta} \equiv \Gamma^{\alpha}{}_{\beta\bar{\gamma}} \overline{dz^{\alpha}} + \Gamma^{\alpha}{}_{\beta\gamma}\, dz^{\gamma} \quad , \tag{2.2}$$

and write Cartan's expressions on this manifold as:

$$d\Gamma^{\alpha}{}_{\beta} + \Gamma^{\alpha}{}_{\delta} \wedge \Gamma^{\delta}{}_{\beta} = \frac{1}{2}\left[R^{\alpha}{}_{\beta\gamma\bar{\delta}}\, dz^{\gamma} \wedge \overline{dz^{\delta}} + R^{\alpha}{}_{\beta\bar{\gamma}\delta}\, d\bar{z}^{\gamma} \wedge dz^{\bar{\delta}}\right]$$

$$= R^{\alpha}{}_{\beta\gamma\bar{\delta}}\, dz^{\gamma} \wedge \overline{dz^{\delta}} \quad , \tag{2.3}$$

where the hat denotes as usually the exterior product between forms and $\{dz^{\alpha}\}$ like $\{dz^{\alpha}\}$ are basis l-forms. Remenbering that the operator d is given by:

$$d = \partial + \bar{\partial}$$

we have

$$d\Gamma^{\alpha}{}_{\beta} = (\partial + \bar{\partial})\Gamma^{\alpha}{}_{\beta} = \Gamma^{\alpha}{}_{\beta\bar{\gamma},\sigma}\, dz^{\sigma} \wedge dz^{\bar{\gamma}} + \Gamma^{\alpha}{}_{\beta\bar{\gamma},\bar{\sigma}}\, \overline{dz^{\sigma}} \wedge \overline{dz^{\gamma}}$$

$$+ \Gamma^{\alpha}{}_{\beta\gamma,\sigma}\, dz^{\bar{\sigma}} \wedge dz^{\gamma} + \Gamma^{\alpha}{}_{\beta\gamma,\sigma}\, dz^{\sigma} \wedge dz^{\gamma} \tag{2.4}$$

where we have used (2.2) we can explicit write the second term in (23) as

$$\Gamma^{\alpha}{}_{\delta} \wedge \Gamma^{\delta}{}_{\beta} = \Gamma^{\alpha}{}_{\beta\bar{\gamma}}\, \overline{dz^{\gamma}} \wedge \Gamma^{\delta}{}_{\beta\bar{\sigma}}\, dz^{\bar{\sigma}} \wedge \Gamma^{\alpha}{}_{\delta\bar{\gamma}}\, \overline{dz^{\gamma}} \wedge \Gamma^{\delta}{}_{\beta\sigma}\, dz^{\sigma}$$

$$+ \Gamma^{\alpha}{}_{\delta\gamma}\, dz^{\gamma} \wedge \Gamma^{\delta}{}_{\beta\bar{\sigma}}\, \overline{dz^{\sigma}} + \Gamma^{\alpha}{}_{\delta\gamma}\, dz^{\gamma} \wedge \Gamma^{\delta}{}_{\beta\sigma}\, dz^{\sigma} \tag{2.5}$$

Substituting (2.4) and (2.5) in (2.3) and comparing both sides of the last expression is easy to deduce for curvature

$$R^{\alpha}{}_{\beta\sigma\bar{\gamma}} = \Gamma^{\alpha}{}_{\beta\bar{\gamma},\sigma} - \Gamma^{\alpha}{}_{\beta\sigma,\bar{\gamma}} + \Gamma^{\alpha}{}_{\delta\bar{\gamma}}\Gamma^{\delta}{}_{\beta\sigma} - \Gamma^{\alpha}{}_{\delta\sigma}\Gamma^{\delta}{}_{\beta\bar{\gamma}} \,, \qquad (2.6)$$

Other similar expressions for $R^{a}{}_{\beta\gamma\delta}$ and $R^{\alpha}{}_{\bar{\beta}\gamma\delta}$ could be deduced in the same way considering other kinds of complex l-forms as:

$$\Gamma^{\alpha}{}_{\bar{\beta}} \equiv \Gamma^{\alpha}{}_{\bar{\beta}\gamma} dz^{\gamma} + \Gamma^{\alpha}{}_{\bar{\beta}\bar{\gamma}} d\overline{z^{\gamma}} \,,$$

$$\Gamma^{\bar{\alpha}}{}_{\beta} \equiv \Gamma^{\bar{\alpha}}{}_{\beta\gamma} dz^{\gamma} + \Gamma^{\bar{\alpha}}{}_{\beta\bar{\gamma}} d\overline{z^{\gamma}} \,,$$

$$\Gamma^{\bar{\alpha}}{}_{\bar{\beta}} \equiv \Gamma^{\bar{\alpha}}{}_{\bar{\beta}\bar{\gamma}} d\overline{z^{\gamma}} + \Gamma^{\bar{\alpha}}{}_{\bar{\beta}\gamma} dz^{\gamma} \,.$$

It's possible to construct complex versions of Einstein equations by taking contractions of curvature expressions like (2.6).

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