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Stability of Stationary Motion of the Symmetrical Top with Rounded Peg

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The stability of stationary motion of the symmetrical top with rounded peg on horizontal plane is studied. On the contrary of what has been appointed, at leat since Routh (1905), it is shown that the possible rolling without sliding stationary motions are not all stable.

Estuda-se a estabilidade do movimento estacionário do pião simétrico de ponta esférica, no plano horizontal. Ao contrário do que tem sido apontado, pelo menos desde Routh (1905), mostra-se que os possiveis movimentos estacionários de rolamento sem deslizamento não são todos estáveis.

1. EQUATION OF MOTION

Consider the symmetrical top with spherical base on the horizontal plane. Let Gz' be the symmetry axis, through the mass center G and the curvature center E (Fig. 1). Let also be: $\underline{m}$ the mass; C and A, the moments of inertia with respect to Gz' and to a barycentric axis normal to Gz', respectively; $\underline{a}$, the radius of the spherical; $\underline{h}$, the eccentricity GE; Gz, the ascending vertical axis.

The reference systems $Gxyz$ and $Gx'y'z'$ have the axes Gy and Gy' in the vertical plane of Gz'. The sense of Gz' is chosen arbitrarily, so that $\underline{h}$ can be positive or negative.

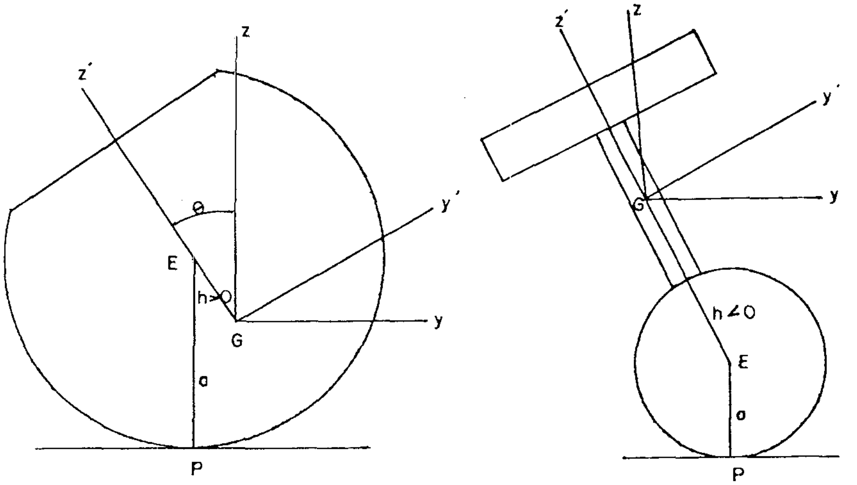

Fig. I

Let F, F' and N be the projections of the reaction of the plane, on the Gx, Gy and Gz axes respectively.

The eulerian components of the rotation of the solid are θ, $\dot{\Psi}$, $\dot{\phi}$, on Gx, Gz and Gz' respectively. The same rotation has the projections

$$p = \dot{\theta}, \quad q = \dot{\Psi}\sin\theta, \quad r = \dot{\phi} + \dot{\Psi}\cos\theta,$$

on the $Gx'y'z'$ axes. The rotation of this $Gx'y'z'$ frame relative to the barycentric reference system of fixed directions has, on the same $Gx'y'z'$ axes, the projections

$$p, \; q, \; r - \dot{\phi} = q \text{ cotan}\theta$$

so that the equation of the motion around the center of mass can be written:

$$\theta = p \, ,$$

$$A\dot{p} - (Aq \text{ cotan}\theta - Cr)q = F' \, (a - h\cos\theta) - Nh\sin \, , \tag{1}$$

$$A\dot{q} + (Aq \text{ cotan}\theta - Cr)p = - F(a\cos\theta - h) \, ,$$

$$C\dot{r} = F a \sin\vartheta \, .$$

Let u, v, w be the projections, on the $Gxyz$ axes, of the velocity of the center of mass with respect to the inertial system of referente. With respect to the barycentric reference system of fixed directions, the $Gxyz$ frame have rotation $\dot{\Psi}$ around Gz, so that the equations of the motion of the center of mass can be written:

$$
\begin{aligned}
m(u' - \Psi v) &= F, \\
m(\dot{v} + \dot{\Psi} u) &= F' , \\
m\dot{w} &= N - mg .
\end{aligned}
\tag{2}
$$

Once the center of mass coordinate on Gz is a-h $\cos\theta$, the last equation yields

$$N = m\left[g + h(\dot{p} \sin\theta + p^2 \cos\theta)\right] . \tag{3}$$

Let also u_p and v_p be the projections, on Gx and Gy, of the velocity of the point of contact P, with respect to the inertial referente system (velocity of sliding); then;

$$
\begin{aligned}
u_p &= u - \left[(a \cos e - h)q - ar \sin\theta\right] , \\
v_p &= v - (a - h \cos\theta)p .
\end{aligned}
\tag{4}
$$

Eliminating F between the last two equations (1) leads to Jellet's integral (cf. [1], page 192), the only integral available for the motion (except for instance in absence on friction or in the rolling without sliding):

$$Aaq \sin\theta + C(a \cos\theta - h)r = K . \tag{5}$$

This integral admits a simple interpretation: it is constant the scalar produt of the angular momentum, by the position vector ofthe point of contact, with origin G.

If we write the equations of the motion around the center of mass using the $Gxyz$ axes,

$$\dot{\theta} = p,$$

$$\dot{U} - \dot{\Psi}V = F'(a - h\cos\theta) - Nh\sin\theta ,$$

$$\dot{V} + \dot{\Psi}U = -F(a - h\cos\theta), \qquad (6)$$

$$W = Fh\sin\theta ,$$

where

$$U = Ap, \quad V = Aq\cos\theta - Cr\sin\theta, \quad W = Aq\sin\theta + Cr\cos\theta ,$$

then we see that the Jellet's integral can also be obtained by eliminating F between the last of equations (1) and the last of equations (6).

2. SLIDING WITHOUT FRICTION

If $F = F' = 0$, the problem can be reduce to quadratures. Indeed, in this case, the motion of the center of mass projected on the plane will be retilinear and uniform. On the other hand, the last three equations (1) can now be replaced by the laws of conservation of angular momentum around Gz' and Gz plus the law of conservation of energy.

It is interesting to note that Jellet's integral will now be superfluos, once it reduces to the combinãtion of the integrals of angular momentum.

A stationary motion ($\underline{\theta}$, $\dot{\Psi}$ and $\underline{r}$ constants), in other words a stationary solution of equations (1), with $F = F' = 0$, is obtained by letting $p = 0$, with the other three variables satisfying the equation:

$$(A\cos\theta\,\dot{\Psi}^2 - Cr\dot{\Psi} - mgh)\sin\theta = 0 , \qquad (7)$$

that is

$$\left[(A - C)\cos\theta\,\dot{\Psi}^2 - C\dot{\phi}\dot{\Psi} - mgh\right]\sin\theta = 0 . \qquad (8)$$

We then see that, once given $\dot{\phi}$ and θ, $\sin\theta \neq 0$, satisfying the condition

$$(C\dot{\phi})^2 + 4\,(A - C)mgh\cos\theta \geq 0 , \qquad (9)$$

then there are two values of Ψ satisfying equation (8). Each of these values will be a root of equation (7), for a convenient $\underline{r}$, that satisfies the condition

$$(Cr)^2 + 4\,Amgh\cos\theta \geq 0\,. \qquad (10)$$

We can also say that, once given $\underline{r}$ and $\underline{\theta}$, $\sin\theta \neq 0$, satisfying condition (10), thereare twovaluesof $\dot{\Psi}$ satisfying equation (7). If these two precessions concerning the same $\underline{r}$ have different absolute values, the one with larger absolute value will be called fast precession, and the other, slow precession.

In order to examine the stability of stationary motion that satisfies equation (7), with $\sin\theta \neq 0$, we take the derivative of thesecond of equations (1) that in this case can be written:

$$(A + mh^2\ \mathrm{sen}^2\theta)\dot{p} = (A\cos\theta\ \dot{\Psi}^2 - Cr\dot{\Psi} - mgh)\sin\theta - mh^2p^2\sin\theta\cos\theta\,. \qquad (11)$$

Taking condition (7) for the stationary motion into account, and neglecting the terms in $p\dot{p}$ or the ones that contains powers of $\underline{p}$ greater than the firçt power, for the disturbed motion, we have:

$$(A + mh^2\sin^2\theta)\ddot{p} = (2A\cos\theta\ \dot{\Psi} - Cr)\ddot{\Psi}\sin\theta - A\sin^2\theta\ \dot{\Psi}^2 p\,.$$

Eliminating $\ddot{\Psi}\sin\theta$ by means of the third of equations (1),with $F = 0$, yields

$$A(A + mh^2\sin^2\theta)\ddot{p} = -[(2A\cos\theta\ \dot{\Psi} - Cr)^2 + (A\sin\theta\ \dot{\Psi})^2]p\,,$$

and then we see that the motion is always stable.

The period of small oscilations of $\underline{p}$ around $p = 0$, which is the same as that of the small oscilations of 0, $\dot{\Psi}$ and $\dot{\phi}$ around their respective values in the stationary motion, will be:

$$T = 2\pi/k$$

$$k^2 = \{[2\cos 0\ \dot{\Psi} - (C/A)r]^2 + (\dot{\Psi}\sin\theta)^2\}/(1 + mh^2\sin^2\theta/A)$$

Once the stationary motion is always stable, fulfilled the e-xistence condition (10), with sin 0 # 0, even for $\sin\theta^2$ arbitrarily small, the discussion for $\sin\theta = 0$ results superfluos. In other words, letting $\theta \to 0$, the motion will also be stable, satisfied (10),

$$(Cr)^2 + 4\,Amgh \geqslant 0 \ ,$$

which is satisfied for any $\underline{r}$, if $h \geqslant 0$. Letting $\theta \to \pi$, the motion will be stable, satisfied the same (10),

$$(Cr)^2 - 4\,Amgh \geqslant 0 \ ,$$

which is satisfied for any $\underline{r}$, if h h 0.

3. EQUATIONS FOR ROLLING WITHOUT SLIDING

Observe initially that using equations (3) and (4) the equations (1) and the first two equations (2), in the general case, can be written

$$\begin{aligned}
&\theta = p\\
&\bar{A}\dot{p} = \left[(\bar{A}\cos\theta - mah\sin^2\theta)\dot{\Psi}^2 - \bar{C}r\dot{\Psi} - mgh\right]\sin\theta -\\
&\quad - mah\sin\theta\, p^2 + m(a - h\cos\theta)(\dot{v}_p + \dot{\Psi}u_p) \ ,\\
&\overline{AC}\sin\theta\,\ddot{\Psi} = -(2\,\overline{AC}\cos\theta\,\dot{\Psi} - C\,\bar{C}r)p - Cm(a\cos\theta - h)(\dot{u}_p - \dot{\Psi}\,v_p) \ ,\\
&d(\overline{AC}\;r^2)/dt = 2A\,mar(\dot{u}_p - \dot{\Psi}\,v_p)\sin\theta \ ,\\
&AC(\dot{u}_p - \dot{\Psi}v_p) = \overline{AC}\;F/m + C\left[(A - C)\,a\cos\theta + Ch\right]rp
\end{aligned} \qquad (12)$$

and a last equation involving $\dot{v}_p + \dot{\Psi}u_p$ and F'/m, by elimination of u and v.

In these equations we take

$$\bar{A} = A + ma^2\sin^2\theta + m(a\cos\theta - h)',$$

$$\bar{C} = C + ma(a - h\cos\theta) ,$$

$$\overline{AC} = AC + Ama^2\sin^2\theta + Cm(a\cos\theta - h)' .$$

Assuming that the top rolls without sliding, also in this case it is possible the separation of variables to study the motion around the center of mass, by means of the first four equations (12), taking

$$u_p = v_p = \dot{u}_p = \dot{v}_p = 0.$$

It is easy to see that, also in this case, the problem can be reduced to quadratures. Indeed, we now have the Jellet's integral (5), while the fourth of equations (12) gives the integral

$$(\overline{AC})^{1/2} r = D , \quad \text{constant,} \tag{13}$$

and also the energy equation will be available.

A stationary motion (8, $\dot{\Psi}$ and r constants), that is a stationary solution of the first four equations (12), is obtained by letting p = 0, with the other three variables satisfying the equation:

$$\left[(\bar{A}\cos\theta - mah\sin^2\theta)\dot{\Psi}^2 - \bar{C}r\dot{\Psi} - mgh\right]\sin 8 = 0 , \tag{14}$$

that is

$$\{\left[(\bar{A} - \bar{C})\cos\theta - mah\sin^2\theta\right]\dot{\Psi}^2 - \bar{C}\dot{\phi}\dot{\Psi} - mgh\}\sin\theta = 0 . \tag{15}$$

We then see that, once given $\dot{\phi}$ and 8, $\sin\theta \neq 0$, satisfying the condition

$$(\bar{C}\dot{\phi})^2 + 4\left[(\bar{A} - \bar{C})\cos\theta - mah\sin^2\theta\right]mgh \geqslant 0 , \tag{16}$$

then there are two values of Ψ satisfying equation (15). Each of these values is one root of equation (14), for a convenient $\underline{r}$ that satisfies the condition

$$(\bar{C}r)^2 + 4(\bar{A}\cos\theta - mah\sin^2\theta)mgh \geq 0 \; . \qquad (17)$$

We can also say that, once given r and θ, $\sin\theta \neq 0$, satisfying condition (17), then thereare two precessions $\dot{\Psi}$ satisfying equation (14). If these two processions concerning the same r have different absolute values, the one with larger absolute value will be called fast precession, and the other, slow precession.

4. THE CASE OF LAGRANGE'S PROBLEM: $a = 0$

The problem in question, taking $a = 0$, is reduced to the classic Lagrange's problem of the solid of revolution with a fixed point. In this case the first four equations (12) are written

$$\dot{\theta} = p,$$

$$(A + mh^2)\dot{p} = \left[(A + mh^2)\cos\theta\dot{\Psi}^2 - Cr\dot{\Psi} - mgh\right]\sin\theta \; ,$$

$$(A + mh^2)\sin\theta\ddot{\Psi} = -\left[2(A + mh^2)\cos\theta\,\dot{\Psi} - Cr\right]p \; ,$$

$$\dot{r} = 0,$$

therefore

$$(A + mh^2)\ddot{p} = \{\left[2(A + mh^2)\cos\theta\dot{\Psi} - Cr\right]\ddot{\Psi} - (A + mh^2)\dot{\Psi}^2\sin\theta p\}\sin\theta,$$

whence

$$\ddot{p} = -\{\left[2\cdot\cos\theta\dot{\Psi} - Cr/(A + mh^2)\right]^2 + (\dot{\Psi}\sin\theta)^2\}p \; ,$$

and then we see that the motion is always stable.

The period of small oscilations of p around $p = 0$, which is also the period of small oscilations of θ, $\dot{\Psi}$ and $\dot{\phi}$ around their respective values in the stationary motion, will be:

$$T = 2\pi/k$$

$$k^2 = \left[2\cos\theta\,\dot\Psi - Cr/(A+mh^2)\right]^2 + (\dot\Psi\ \sin\theta)^2 .$$

Once the stationary motion is always stable, satisfied the existence condition (17), for $\sin\theta \neq 0$, even for $\sin^2\theta$ arbitrarily small, the discussion for $\sin\theta = 0$ results superfluos. In other words, letting $\theta \to 0$, the stationary motion is always stable, if

$$(Cr)^2 + 4(A + mh^2)mgh \geq 0 ,$$

which is satisfied for any $\underline{r}$, if $h \geq 0$. Letting $\theta \to \pi$, the motion will be stable, satisfied the same (17),

$$(Cr)^2 - 4(A + mh^2)\ mgh \geq 0 ,$$

which is satisfied for any $\underline{r}$, if $h \leq 0$.

5. THE GENERAL CASE OF ROLLING WITHOUT SLIDING: a ≠ 0

Let us suppose now that $a \neq 0$, and, to simplify, let be $h/a=\lambda$ and replace the quotients A/ma^2, C/ma^2, $\bar A/ma^2$, $\bar C/ma^2$ and $\overline{AC}/ma^2$, by A, C, $\bar A$, $\bar C$ and $\overline{AC}$ respectively. Let us also define $\bar g = g/a$. Then the second of equations (12) will be written:

$$A\dot p = \left[(\bar A\cos\theta - A\sin^2\theta)\Psi^2 - Cr\Psi - \bar g\lambda\right]\sin\theta - \lambda\sin\theta\, p^2. \qquad (18)$$

Observe now that, denoting $\bar C' = d\bar C/d\theta$ etc, and taking into account the integral (13) we have:

$$\bar C' = \lambda\sin\theta,\quad A' = 2\lambda\sin\theta,\quad (\bar A\cos\theta - \lambda\sin^2\theta)' = -A'\sin\theta,$$

$$r' = -(1/2)\ (\overline{AC}'/\overline{AC})r ,$$

therefore

$$(Cr)' = -\left[(1/2)(\overline{AC}'/\sin\theta)\bar C - \overline{AC}\lambda\right]r\ \sin\theta/\overline{AC}$$

Also note that it is possible to establish the identities:

$(1/2)(\overline{AC}'/\sin\theta) = \bar{A}\cos\theta - \lambda\sin^2\theta - \bar{C}\chi$;

$\overline{AC}\cos = (\bar{C} - \chi\cos\theta)(\bar{A}\cos\theta - \lambda\sin^2\theta) - \bar{C}\chi\sin^2 8,$

where we denote

$\chi = \cos\theta - \lambda$

Then we will perform the derivative of the second of equations (12), that if of equation (18), taking into account the condition for stationary motion.

$$(\bar{A}\cos\theta - \lambda\sin^2\theta)\dot{\Psi}^2 - \bar{C}r\dot{\Psi} - \bar{g}\lambda = 0 \ , \qquad (19)$$

which requires

$$(\bar{C}r)^2 + 4(\bar{A}\cos\theta - \lambda\sin^2\theta)\bar{g}\lambda \geq 0 \ . \qquad (20)$$

Neglecting the terms in $p\dot{p}$ or those that contains powers of $\underline{p}$ greater than the first power, and eliminating $\ddot{\Psi}$ by means of the third of equations (12) with $\dot{u}_p - \dot{\Psi}v_p = 0$, we obtain:

$$\overline{AC}\,\bar{A}\,\ddot{p}/p = -C\xi^2 - \{2(\alpha-\bar{C}\chi)\xi + \overline{AC}\,\bar{A}\dot{\Psi} - [(\alpha-\bar{C}\chi)\bar{C} - \overline{AC}\lambda]r\}\dot{\Psi}\sin^2\theta, \qquad (21)$$

where α and ξ are defined by

$\bar{A}\cos\theta - \lambda\sin^2\theta = \alpha$, $2\alpha\dot{\Psi} - Cr = \xi$

Hence we can set: that, if $\sin^2\theta$ is sufficiently small, the motion will always be stable, satisfied (20), which is satisfied for any r, if $0 = 0$, $A \geq 0$, or $0 = \pi$, $\lambda \leq 0$. The period of small oscilations of $\underline{p}$ around $p = 0$, or of 0, $\dot{\Psi}$ aand $\underline{r}$ or $\dot{\phi}$ around their respective values in the stationary motion, will be:

$$T = 2\pi/k \quad ; \quad k^2 = C\,\xi^2/\overline{AC}\,\bar{A} \ .$$

Consider again the general case. Eliminating $\dot{\Psi}$ by means of the same preceding transformation, defining

$$\bar{\xi} = \xi/\bar{C}r \quad ,$$

it results from (21):

$$(4\ \overline{AC}\ \bar{A}\ \alpha^2/\bar{C}r^2)\ \ddot{p}/p = Q(\bar{\xi});$$

$$Q(\bar{\xi}) = -\ 4\ C\ \bar{C}\ \alpha^2\ \bar{\xi}^2\ -$$

$$-\ \{4\alpha(\alpha-\bar{C}\chi)\bar{C}\bar{\xi} + \overline{AC}\ \bar{A}\ \bar{C}(\bar{\xi} + 1)-2[(\alpha - \bar{C}\chi)\bar{C} - \overline{AC}\lambda]\alpha\}(\bar{\xi}+1)\ \sin^2\theta$$

Note that, satisfied condition (20), the roots of equation (19) can be written:

$$\bar{\Psi} = (\bar{C}r/2\alpha)\ \{1 \pm [1 + 4\ \alpha\bar{g}\lambda/(\bar{C}r)^2]^{1/2}\}\ ;$$

to the plus sign it corresponds the fast precession, and to the minus the slow one.

It is easy to see that, by means of the preceding transformation

$$(2\alpha\Psi - \bar{C}r)/\bar{C}r = \bar{\xi}\ ,$$

what we must do, to decide about the stability of stationary motion, is to examine the sign of the trinominal $Q(\bar{\xi})$, for the values that $\bar{\xi}$ can take, in each case, taking

$$\bar{\xi} = [1 + 4\alpha\bar{g}\lambda/(\bar{C}r)^2]^{1/2}\ , \text{ for the fast precession;}$$

$$\bar{\xi} = -\ [1 + 4\ \alpha\bar{g}\lambda/(\bar{C}r)^2]^{1/2}\ , \text{ for the slow precession.}$$

If $\alpha\lambda > 0$, then

$1 < \bar{\xi} < \infty$, for fast precession;

$-\infty < \bar{\xi} < -1$, for slow precession, If $\alpha\lambda < 0$, then

$0 < \bar{\xi} < 1$, for fast precession,

$-1 < \bar{\xi} < 0$, for slow precession.

It is easy to see that we always have $\bar{C} > 0$, with the center of mass of the solid above the horizontal plane ($1 - \lambda\cos\theta > 0$), hence, in this case, the stationary motion will be stable, if $Q(\bar{\xi}) < 0$; and unstable, on the contrary.

According to this criterion, first it is possible to show that the motion is always stable if $(\bar{C}r)^2$ is sufficiently large. Indeed, it is immediately seen that $Q(-1)<0$, so that the slow precession is stable for $(\bar{C}r)^2$ sufficiently large. On the other hand we can obtain:

$$Q(1)=-4\,\overline{AC}\{[C+\sin^2\theta+(1-\lambda\cos\theta)\cos^2\theta]A+[\chi^2+(1-\lambda\cos\theta)\sin^2\theta]C+(1-\lambda\cos\theta)^3\},$$

and then we also see that $Q(1) < 0$ for any θ, whatever are the structural pararneters of the solid, A, C and λ, with the center of mass above the horizontal plane, so that, in this case, the fast precession will also be always stable, for $(\bar{C}r)^2$ sufficiently large.

Hence we also see that, once, if A = 0, the slow precession corresponds to $\bar{\xi} = -1$ (in this case, null precession), and the fast precession corresponds to $\bar{\xi} = 1$, then, if $\lambda = 0$, the stationary precession (slow or fast) will always be stable. In particular, the stationary precessions of a billiard ball (A = 0, A = C) will always be stable.

What remains to be proved is that, if $(\bar{C}r)^2$ is not sufficiently large, although it is so that condition (20) is satisfied, then there are pararneters A, C and A, defining the structure of the solid, in such a way that, because of the nutation θ, there will be unstable stationary precessions. In other words, what remains to be proved is that (20) is an existence condition for stationary motion, not always sufficient for its stability.

With this aim, first note that

$$Q(0) = \{2[(\alpha - C\chi)\bar{C} - \overline{AC}\,\lambda]\alpha - \overline{AC}\,\bar{A}\,\bar{C}\}\sin^2\theta\ ,$$

or

$$Q(0)/\sin^2\theta = 2\bar{C}(A\cos\theta - C\chi)\left[A\cos\theta + \chi(1-\lambda\cos\theta)\right] -$$

$$-\left[(\bar{C} - \chi\cos\theta)A + C\chi^2\right]\left[(C + 2\lambda\cos\theta)A + (\sin^2\theta + \chi^2)\bar{C} + 2\lambda\chi(1-\lambda\cos\theta)\right],$$

a trinomial of second degree in A that has as coefficient of A the trinominal

$$P(\tilde{C}) = -\left[\tilde{C}^2 - \chi\cos\theta\,\tilde{C} - 2\chi\cos^3\theta\right]; \quad \tilde{C} = C + \sin^2\theta.$$

Hence it results that, if $\lambda < 0$ and $\cos\theta > 0$ (which implies **that** $\chi = \cos\theta - \lambda > 0$, $\alpha = A\cos\theta + \chi(1-\lambda\cos\theta) > 0$, hence $\alpha\lambda < 0$), then $P(\tilde{C}) > 0$, if

$$C + \sin^2\theta < (1/2)\{\chi + [(\chi + 8\cos\theta)\chi]^{1/2}\}\cos\theta,$$

therefore $Q(0) > 0$, for sufficiently large A. Consequently, $Q(\bar{\xi})$ will not be negative in the whole interval $-1 < \bar{\xi} < 0$, neither in the whole $0 < \bar{\xi} < 1$, as it would be necessary in order that the stationary precessions (slow or fast) were always stable.

Observe now that the coefficient of $\bar{\xi}^2$ in $Q(\bar{\xi})$ can be written

$$-\{4C\alpha^2 + [4\alpha(\alpha - \bar{C}\chi) + \overline{AC}\,\bar{A}]\sin^2\theta\}\bar{C} =$$

$$-\overline{AC}\,\bar{C}\left[(4\cos^2\theta + \sin^2\theta)A + (2\chi\cos\theta + \sin^2\theta)^2 + \chi^2\sin^2\theta\right],$$

so that it is always negative, hence $Q(\pm\infty) < 0$. Once it has been verified that &(+ 1) < 0, it is seen that, if $a), > 0$, then, if unstable precessions exist, only there exist them slow or only fast.

Note also that we can obtain

$$Q'(-1)/2\alpha = J(C)A + \left[(4\bar{C}\cos\theta - \lambda\sin^2\theta)\chi + \bar{C}\sin^2\theta\right]C\chi ,$$

$$J(C) = 4\cos\theta\, C^2 + \left[(3\cos^2\theta + 1)\chi + 6\cos\theta\sin^2\theta\right]C +$$

$$+ \left[3\cos\theta - (2\cos^2\theta + 1)\lambda\right]\sin^2\theta,$$

hence, if

$0 < \cos\theta < 1$, $\lambda > 3\cos\theta/(2\cos^2\theta + 1)$

it results

$\alpha\lambda > 0$, $Q'(-1) < 0$,

for sufficiently small C and sufficiently large A.

On the other hand, for the discriminant A of the trinomíal $Q(\bar{\xi})$ we can write:

$$\Delta/4\alpha^2 \sin^2\theta = [3\bar{C}(\alpha-\bar{C}\chi)-\overline{AC}\,\lambda]^2 \sin^2\theta - 4C\bar{C}\{\overline{AC}\,\bar{A}\,\bar{C} - 2[(\alpha-\bar{C}\chi)\bar{C} - \overline{AC}\,\lambda]\alpha\},$$

hence we see that A ⩾ 0, for sufficiently small C

Therefore it follows that, also in the case $\alpha\lambda > 0$, unstable slow precessions exist.

6. REMARK ON ROUTH'S WORK

Let us write the energy equation for the symmetrical top with rounded peg that rolls without sliding on the horizontal plane. We have:

$$Ap^2 + Aq^2 + Cr^2 + m(u^2 + v^2 + w^2) + 2mg(a - h\cos\theta) = 2E,$$

constant.

Eliminating q and r by means of the integral (5) and (13) yields

$\bar{A}p^2 = R(\theta)$,

$R(\theta) = R_1(\theta)/S_1(\theta) + R_2(\theta)$,

$$R_1(\theta) = -\{D^2/(\overline{AC}-AC) - 2KDm(a\cos\theta-h)\overline{AC}^{1/2} + K^2m[A + m(a\cos\theta - h)^2]\},$$

$S_1(\theta) = A^2 ma^2 \sin^2\theta,$

$R_2(\theta) = 2[E - mg(a - h\cos\theta)]$,

thus the problem is reduced to one degree of freedom. As we have seen, the motion will be stationary, for $\theta = \bar{\theta}$, if $R(\bar{\theta}) = R'(\bar{\theta}) = 0$, and stable, if $R''(\bar{\theta}) < 0$, a condition not always satisfied, as we have shown.

Naturally the derivative of (22) with respect to time, once divided by p,

$$\bar{A}\dot{p} = (1/2)\, R'(\theta) - mah \sin^2\theta\, p^2 ,$$

is no other than the second of equations (12), that have been used by us as the base for discussion, if integrals (5) and (13) are there substituted.

The mistake made by Routh ([1], page 193 and the following) consists in treating the right member of (22), once multiplied by $\sin^2\theta$ and substituting $x = \cos\theta$, $1 - x^2 = \sin^2\theta$, as it would be a third degree polynomial in x, which is what we obtain in the reduction of Lagrange's problem.

The only further explicit reference we have found concerning the matter is one made by Hughenholtz[2], about which we cannot say anything, once only the (wrong) result is appointed, which were obtained after a rather long calculation, according to the author.

7. CONCLUSION

We have shown the possible rolling without sliding stationary motions of the symmetrical topwith rounded peg on horizontal plane are not all stable. In other words, condition (20), which is verified for any r if $\alpha\lambda > 0$, is an existence condition for stationary motion, not always sufficient for its stability.

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