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Classical Chromodynamics, a Geometrical Approach

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It is shown how Wong's equations for point-particles with non--abelian charge interacting with the Yang-Mills field, can be derived in a geometrical way without resorting to field theory. Equations of motion for extended objects are derived using the same method. A short discussion of the point-particle case shows the appearance of non-local features in the Yang-Mills-Wong equations.

Mostra-se como as equações de Wong para partículas puntiformes com carga **não-abeliana'interagindo** com o campo de Yang-Mills, podem ser obtidas por considerações geométricas sem passar pela teoria dos campos. obtem-se também equações para cordas não-abelianas pelo mesmo método. O caso de partículas puntiformes mostra certos aspectos não-locais nas e-quações de Yang-Mills-Wong.

1. INTRODUCTION

The electromagnetic interaction of charged matter is currently described as mediated by an abelian gauge field. This electromagnetic field obeys the Maxwell equations with a source term depending on the charged matter, while the dynamics of the matter itself is determined by some minimal interaction principle.

The description of matter can be considered in two different ways. In the first approach matter is identified with geometric entities in space-time. Usually one introduces the concept of pointlike[2] (in 3-space) particles endowed with mass and carge but extended objects such as strings, bubbles or bags can be considered also. The Lorentz force describes the interaction with the field, but the field entering the usual expression of this force is not the sum of the incoming and the retarded field, which diverges on the worldline of the point-particle. The renormalization procedure of P.A.M. Dirac[1] can handle this problem in a relatively satisfactory way as long as one considers only motions which are asymptotically free. The role of the asymptotic condition in the renormalization procedure was stressed by R. Haag so that, ab initio, the renormalized equations of motion will not admit any purely electromagnetic bound state solutions.

In the second approach matter also is described by a field whose dynamics is governed by field equations. This is classical field theory. In a final stage both the electromagnetic field and the matter description should be quantized.

In a similar way the strong inte-action of coloured matter is believed to be mediated by a non-abelian gauge field. This gluonic field obeys the Yang-Mills equations with a source term constructed from the coloured matter. In principle the coloured matter can also be described in the two different ways mentioned above but in practice the main work was done within the second approach. A large number of solutions of the sourceless Yang-Mills equations ha; been found both in Euclidean and Minkowsky space-time[3]. The analogue of the Lorentz equation for pointlike coloured matter has been derived by S.K.Wong[4] starting from field equations. The classical non-abelian charge depends on the worldline parameter and, as shown by H.Arodź[5], it can be interpreted as the expectation value of the corresponding quantum operator in coherent states.

In this work we show how these equations can be obtained in a geometrical way without resorting to field theory. The idea is not new and goes back to Kaluza-Klein via the work of R. Utiyama, R. Kerner, A. Trautman, Y.M.Cho[6] and others. It should be noted however that, except for the work of R.Kerner*, emphasis was generally put on the field equations and not on the dynamics of matter.

We are also able to generalize the Wong equations for extended objects in a manifestly parametrization invariant way. Our equationsare different from those postulated by A.P.Balachandran and coworkers[7] which appear to be parametrization invariant (a geometric necessity) only if certain constraints are satisfied by the gauge field (a geometric entity). In the case of pointlike particles it is shown that the formal solution of the motion of the non-abelian charge leads to a generally non-local Lorentz type equation of motion. Naturally at this stage the theory has the same drawbacks as the usual formulation of the Maxwell-Lorentz theory of classical electrodynamics. Due to the inherent non-linearity of the theory it does not seem to be straightforward to generalize Dirac's analysis to the non-abeliãn case. The perturbative approach of W. Drechsler and A. Rosenblum[13] based on the use of Riesz potentials probably does not allow for bound states.

Concerning the stringlike particles, we are led to a non-linear hyperbolic differential equation describing the interaction of a chiral field, defined on the string, with the gauge field. In the abelian case this equation becomes linear and the Cauchy problem is easily solved. When the field is pure gauge, the inverse scattering method can be used. The genuine interaction case is the object of further study.

2. THE GEOMETRY

The usual geometrical setting of gauge theories[8] is based on a principal fibre bundle P over space-time M with structural group G and projection $\pi : P \to M$. A local gauge is given by an open subset U of M and a diffeomorphism $\phi: U \times G \to \pi^{-1}(U)$: $(x,h) \to p = \phi(x,h) = \phi_x(h)$ such that $\pi(\phi(x,h)) = x$. When x belongs to the intersection of two local gauges (U,ϕ) and (U',ϕ') one has the patching condition: if $p = \phi_x(h) =$

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* Unfortunatelly this paper contains some calculational errors so that the correct Wong equations are not obtained.

$= \phi'_x(h')$ then $h' = t(x).h$ where $t(x) = (\phi'_x)^{-1}.\phi_x$ belongs to G. Fibre coordinates (p^A), $A = 1,2,\ldots,4+N$ are given by the coordinates of x and h: (x^i), $i = 0,1,2,3$ and (h^a), $\alpha = 1,2,\ldots,N$. A right action of the group G on P can be defined in a local gauge by $R_g : P \to P : p = \phi(x,h) \to R_g(p) = \phi(x,h.g)$. Let ε_α be a basis of the Lie algebra $A(G)$ of G with $|\varepsilon_\alpha, \varepsilon_\beta| = c^\lambda_{\alpha\beta}\,\varepsilon_\lambda$, exponentiation yields $g_\alpha(t) = \exp(t\varepsilon_\alpha)$ and one defines the fundamental vector fields

$$\vec{e}_\alpha(p) = \frac{d}{dt} R_{g_\alpha(t)}(p)\Big|_{t=0}$$

which obey $[\vec{e}_\alpha\, \vec{e}_\beta] = c^A_{\alpha\beta}\, \vec{e}_\lambda$.

The principal fibre bundle $P(M,G,\pi)$ is endowed with a connection defined by a one-form on P with values in $A(G)$: $\omega = \omega^a \varepsilon_a$ such that $\omega^\alpha(\vec{e}_B) = \delta^a_B$ and $R^*_g\omega = Ad(g)\omega$ where R^*_g is the pull-backmapof R_g and where $Ad(g)$ is the adjoint representation of G in $A(G)$. For sake of definiteness we shall work with the adjoint representation of G and of $A(G)$; the basis elements of the Lie algebra ε_α are then represented by rnatrices with elements given by $(\varepsilon_a)^A_\mu = c^A_{\alpha\mu}$ and $(Ad(g))^\alpha_\beta$ will be denoted by $(g)^a_\beta$. The above requirements on ω lead to the following expression in a local gauge (U,ϕ) : $\omega(x,h) = h^{-1}\,\alpha(x)h + h^{-1}dh$ where $\alpha(x) = A.(x)\, dx^i$ is the Lie algebra valued one-form on M giving the gauge potential.

A gauge transformation is a bundle automorphism reducing to the identity on M. It is given in a local gauge (U,ϕ) by: $\tau : P \to P : p = \phi(x,h) \to p' = \tau(p) = \phi(x,g_\tau(x).h)$. The pull-back of τ defines a new connection which in the same local gauge is given by $\omega'(x,h) = h^{-1}\,\alpha'(x)h + h^{-1}dh$ where $a'(x) = g_\tau(x)\,\alpha(x)\,g_\tau^{-1}(x) - dg_\tau(x)\,g_\tau^{-1}(x)$.

Instead of defining fields for the matter description, we will stay in the $4+N$ dimensional fibre bundle and describe matter by worldlines and worldsheets in P:

a) point-particles $p^A = z^A(u)$, $u \in [-\infty, +\infty]$.

or $x^i = y^i(u)$ and $h^\alpha = g^\alpha(u)$

b) string-particles $p^A = z^A(\xi^0, \xi^1), \xi^0 \in [-\infty, +\infty]$
and $\xi^1 \in [0, \pi]$
or $x^i = y^i(\xi)$ and $h^\alpha = g^\alpha(\xi)$

In P one constructs a metric from the metric in M and from the bi-invariant metric in the Lie group:

$$G_{..} = g_{ij}\, dx^i \otimes dx^j + a^2\, \eta_{\alpha\beta}\, \omega^\alpha(x,h) \otimes \omega^\beta(x,h)$$

where $\eta_{\alpha\beta} = c^\lambda_{\alpha\mu}\, c^\mu_{\beta\lambda}$ is negative definite for compact semisimple groups.

The adapted basis of one-forms $\theta^A(x,h)$ is obviously given by $(dx)^i$ and (ω^α) whose dual basis in tangent space is given by $\vec{e}_\alpha(h) = L^\mu_\alpha(h,1)\, \vec{\partial}_\mu$

$$\vec{e}_i(x,h) = \vec{\partial}_i - \omega^\alpha_i(x,h)\vec{e}_\alpha(h)$$

where

$$L^\mu_\alpha(g,h) = \frac{\partial(h.g)^\mu}{\partial g^\alpha}$$

The Lie brackets of this $(4+N)$-bein are obtained as

$$[\vec{e}_i, \vec{e}_j] = -(h^{-1})^\alpha_\mu\, F^\mu_{ij}(x)\vec{e}_\alpha$$

$$[\vec{e}_i, \vec{e}_\alpha] = 0$$

$$[\vec{e}_\alpha, \vec{e}_\beta] = c^\lambda_{\alpha\beta}\, \vec{e}_\lambda$$

with, in matrix notation, $F_{ij} = \partial_i A_j - \partial_j A_i + [A_i, A_j]$

or $\Phi = 1/2\, F_{ij}\, dx^i \wedge dx^j = d\alpha + \alpha \wedge \alpha$

This metric is invariant under the right action R_g which implies that the fundamental vector fields $\vec{e}_\alpha$ are Killing vector fields. Furthermorr, since a^2 is constant, these Killing vector fields generate geodesic displacements.

The associates metric connection is given by the connection one-forms:

$$\Gamma^{\alpha}_{\beta} = 1/2\ c^{\alpha}_{\mu\beta}\ \omega^{\mu}$$

$$\Gamma^{\alpha}_{b} = -\ 1/2\ (h^{-1})^{\alpha}_{\mu}\ F^{\mu}_{ib}(x)\ dx^{i}$$

$$\Gamma^{a}_{\beta} = -\ 1/2\ a^{2}\ \eta_{\alpha\beta}(h^{-1})^{\beta}_{\mu}\ F^{\mu a}_{i}(x) dx^{i}$$

$$\Gamma^{a}_{b} = \{{a \atop ib}\}\ dx^{i} - 1/2\ a^{2}\ \eta_{\alpha\beta}\ (h^{-1})^{\beta}_{\mu}\ F^{\mu a}_{b}(x)\ \omega^{\alpha}$$

where $\{{a \atop ib}\}$ are the Christoffel symbols in M,

After some caiculations one obtains the scalar curvature:

$$R = R_{M}(x) + 1/4\ a^{2}\ \mathrm{Tr}(F_{ij}(x) F^{ij}(x)) + N/4a^{2}$$

which appear to be independent of the particular point chosen in the fibre over x.

$R_{M}(x)$ is the scalar curvature in the base space M and we have adopted the conventions of Y.Choquet-Bruhat et *al.*[9]. The invariant volume element factorises in the adapted basis:

$$dV = [\det G_{..}]^{1/2}\ dx^{0} \wedge dx^{1} \wedge dx^{2} \wedge dx^{3} \wedge \omega^{1} \wedge \ldots \wedge \omega^{N}$$

$$= \sqrt{\Delta}\ \ dx^{0} \wedge dx^{1} \wedge dx^{2} \wedge dx^{3}) \wedge (a^{N}\ \mu^{1} \wedge \ldots \wedge \mu^{N})$$

with $\Delta(x) = [\det\ \ g_{ij}(x)]$

and $\mu^{\alpha}\varepsilon_{\alpha} = \mu = h^{-1} dh$ is the Maurer-Cartan form.

3. THE ACTION PRINCIPLE AND THE EQUATIONS OF MOTION

The total action will be given by the field action $I_{F} = A\int dV\ R$ and the rnatter action which consists in a sum of terms of the form $I_{\text{point}} = -mc \int d\ell$ with

$$d\ell = [G(\vec{v}, \vec{v})]^{1/2}\ du$$

$$\vec{v}(u) = \dot{z}^{A}(u)\ \vec{\partial}_{A}$$

being the tangent vector to the wordline, and $I_{\text{string}} = - nc \int dS$, where $dS = \sqrt{M} \; d\xi^0 \quad d\xi^1$

$$M(\xi) = [\det m_{ab}(\xi)] \;,$$

$$m_{ab} = G(\vec{v}_a, \vec{v}_b)$$

and

$$\vec{v}_a(\xi) = z^A{}_{,a}(\xi) \; \vec{\partial}_A$$

are the tangents (a, b,... = $0,1$) to the worldsheet.

The equation of motion are obtained from the action principle:

$$\delta(I_F + I_{\text{point}} + I_{\text{string}}) = 0$$

for variations of the fields and of the trajectories which vanish atthe boundary of the integration domain except for the open string for which $\delta z^A(\xi^0, \xi^1)$ is arbitrary when $\xi^1 = 0$ or v.

In the field action, the integration over the group manifold can be done and one obtains

$$I_F = A \; V_G \quad \int \quad d^4x \; \sqrt{\Delta} \quad R(x)$$

where V_G is the group volume times a^N

In order to obtain the usual Einstein-Maxwell-Yang-Mills equations, a^2 has to be positive and

$$a^2 = \frac{16\pi\kappa}{c^3} \; \frac{e^2}{4\pi c} \quad ,$$

where κ is the gravitational constant and where e is the unit of charge such that $F^{\alpha}_{ij}(x) = e \; F^{\alpha}_{ij}(x)$ has the dimensions charge/(length)^2.

The energy-momentum tensor of the Yang-Mills field is

$$T^{ij}_{YM}(x) = - \frac{1}{4\pi c} \; \eta_{\alpha\beta} \left(\frac{1}{4} \; F^{\alpha pq} \; F^{\beta}_{\sim\sim} \; g^{ij} \; - \; F^{\alpha ip} \; F^{\beta j}_{p}\right)$$

and

$$\delta I_F = - AV_G \left\{\int d^4x \sqrt{\Delta} \left[S_M^{ij} - \frac{N}{8a^2} g^{ij} + \frac{8\pi\kappa}{c^3} T^{ij}_{YM}\right] \delta g_{ij}\right.$$

$$\left. + \int d^4x \sqrt{\Delta} \frac{a^2}{e} \eta_{\alpha\beta} (\nabla_i F^{\alpha ij}) \delta A^\beta_j \right\}$$

where S_M^{ij} is the Einstein tensor in M and

$$\nabla_i F^{ij} = \partial_i F^{ij} + \{{}^{i}_{ip}\} F^{pj} + \{{}^{j}_{ik}\} F^{ik} + [A_i, F^{ij}]$$

is the covariant divergence of F.

For the point particle we introduce the (gauge-invariant) components of the velocity

$$v^i(u) = dx^i(\vec{v}) \text{ and } v^\alpha(u) = \omega^\alpha(\vec{v}) \ .$$

The first ones are simply $v^i = \dot{y}^i$, while the second ones are given in matrix form by

$$V(u) = g^{-1}(u)\ A_i(y(u))\ g(u)\ \dot{y}^i(u) + g^{-1}(u)\ \dot{g}(u)$$

In a manifestly parametrization invariant form one obtains:

$$I_{\text{point}} = mc \int du\ \dot{\ell}\ \left[\frac{1}{\dot{\ell}} \frac{d}{du} \frac{\dot{z}^A}{\dot{\ell}} + \Gamma^A_{MN} \frac{\dot{z}^M\ \dot{z}^N}{(\dot{\ell})^2}\right] G_{AB}\ \delta z^B$$

$$- 1/2 \int d^4x \sqrt{\Delta}\ T^{ij}_{\text{point}}\ \delta g_{ij}$$

$$- \frac{e}{4\pi c} \int d^4x \sqrt{\Delta}\ \eta_{\alpha\beta}\ J^{\alpha j}_{\text{point}}\ \delta A^\beta_j$$

where $\dot{\ell} = \frac{d\ell}{du}$, and with energy-momentum tensor and current given by:

$$T^{ij}_{\text{point}}(x) = \frac{1}{\sqrt{\Delta}} \int du\ \dot{\ell}\ mc \frac{\dot{y}^i\ \dot{y}^j}{(\dot{\ell})^2} \delta^4 (x - y(u)) \ .$$

$$J^j_{\text{point}}(x) = \frac{1}{\sqrt{\Delta}} \int du\ \dot{\ell}\ c\ I(u) \frac{\dot{y}^j}{\dot{\ell}} \delta^4 (x - y(u)) \ .$$

The unit of charge for a point particle is $q = mc^2a/e$ and depends on its (bare) mass m.

We introduce the intrinsic charge $Q(u) = -q \frac{a\ V(u)}{\dot{\ell}}$ and the effective charge

$$I(u) = g(u)\ Q(u)\ g^{-1}(u)\ .$$

In a similar way we define for the string-particle

$$v^i_a(\xi) = dx^i(\vec{v}_a) = y^i{}_{,a}(\xi)$$

and $v^{\alpha}_a(\xi) = \omega^{\alpha}(\vec{v}_a)$ given by the matrix

$$V_a(\xi) = g^{-1}(\xi)\ A_i(y(\xi))\ g(\xi)\ y^i{}_{,a}(\xi) + g^{-1}(\xi)\ \ g_{,a}(\xi)$$

The variation of the string action leads to

$$\delta I_{\text{string}} =$$

$$nc \int d\xi^0 \quad d\xi^1\ \sqrt{M} \left(\frac{1}{\sqrt{M}}\ \ \partial_a(\sqrt{M}\ m^{ab}\ z^A{}_{,b}) + \Gamma^A_{MN}\ m^{ab}\ z^M{}_{,a}\ \ z^N{}_{,b}\right) G_{AB}\ \delta z^B$$

$$- nc \int d\xi^0 \wedge d\xi^1\ \ \partial_b(\sqrt{M}\ m^{ab}\ z^A{}_{,a}\ G_{AB}\ \delta z^B)$$

$$- \frac{1}{2} \int d^4x\ \sqrt{\Delta}\ T^{ij}_{\text{string}}\ \ \delta g_{ij}$$

$$- \frac{e}{4\pi c} \int d^4x\ \sqrt{\Delta}\ \eta_{\alpha\beta}\ J^{\alpha j}_{\text{string}}\ \ \delta A^{\beta}_{,j}$$

The charge unit of a string particle is $q' = \frac{n\ c^2\ a^2}{}$ and depends on the (bare) tension *n*.

The intrinsic and effective charge currents on the string are defined by $Q_a(\xi) = -q'\ V_a(\xi)$ and $I_a(\xi) = g^{-1}(\xi)\ Q_a(\xi)\ g(\xi)$.

Energy-momentum and current in space-time are

$$T^{ij}_{\text{string}}(x) = \frac{1}{\sqrt{\Delta}} \int d\xi^0 \wedge d\xi^1\ \sqrt{M}\ m^{ab}\ y^i{}_{,a}\ y^j{}_{,b}\ \ \delta^4(x - y(\xi))$$

$$J^j_{\text{string}}(x) = \frac{1}{\sqrt{\Delta}} \int d\xi^0 \quad d\xi^1\ \ \sqrt{M}\ m^{ab}\ c\ I_a(\xi)\ y^j{}_{,b}\ \delta^4(x - y(\xi)).$$

Fixing the constant A by A $V_G = c^3/16\pi\kappa$ we obtain the following set of equations

a) the Einstein-Maxwell-Yang-Mills equations:

$$S^{ij}_{M} - \frac{N}{8a^2} + \frac{8\pi\kappa}{c^3}\left(T^{ij}_{YM} + T^{ij}_{\text{point}} + T^{ij}_{\text{string}}\right) = 0 \tag{1}$$

$$\nabla_i F^{ij} = \frac{4\pi}{c}\left\{J^j_{\text{point}} + J^j_{\text{string}}\right\} \tag{2}$$

b) the Lorentz-Wong equations:

for the point-particle

$$\frac{1}{\dot{\ell}}\frac{d}{du}\left(\frac{\dot{z}^A}{\dot{\ell}}\right) + \Gamma^A_{MN}\frac{\dot{z}^M\,\dot{z}^N}{(\dot{\ell})^2} = 0 \tag{3}$$

and for the string

$$\frac{1}{\sqrt{M}}\,\partial_a\left(\sqrt{M}\,m^{ab}\,z^A{}_{,b}\right) + \Gamma^A_{MN}\,m^{ab}\,z^M{}_{,a}\,z^N{}_{,b} = 0 \tag{3'}$$

with the boundary condition

$$v^{1a}\,z^A{}_{,a} = 0 \quad \text{when } \xi^1 = 0 \text{ or } \pi\,. \tag{BC}$$

The Lorentz-Wong equations are more transparent when written in the adapted basis.

For the point particle one obtains

$$\frac{1}{\dot{\ell}}\,\frac{d}{du}\,\frac{V(u)}{\dot{\ell}} = 0 \tag{4a}$$

$$m\,c^2\left(\frac{1}{\dot{\ell}}\frac{d}{du}\frac{\dot{y}^i}{\dot{\ell}} + \left\{{i \atop pq}\right\}\frac{\dot{y}^p\,\dot{y}^q}{(\dot{\ell})}\right) = \mathrm{Tr}\left(I(u)\;F^i_j(y(u))\right)\frac{\dot{y}^j}{\dot{\ell}} \tag{4b}$$

Equation (4a) expresses the conservation of the intrinsic charge and could also be expressed as the covariant conservation of the effective charge(*)

$$\frac{d}{du}\,I(u) + \dot{y}^i(u)\;\left[A_i(y(u)),I(u)\right] = 0 \tag{5}$$

(*) $Q(u)$ is gauge invariant, while $I(u)$ transforrns under the adjoint representation.

In the same way one obtains the string equations:

$$\frac{1}{\sqrt{M}}\,\partial_a(\sqrt{M}\,m^{ab}\,V_b) = 0 \tag{4'a}$$

$$nc^2\left(\frac{1}{\sqrt{M}}\,\partial_a(\sqrt{M}\,m^{ab}\,y^i_{,b}) + \left\{{i \atop pq}\right\} m^{ab}\,y^p_{,a}\,y^q_{,b}\right) =$$

$$\mathrm{Tr}(I_a(\xi)F^i_j(y(\xi)))\;m^{ab}\,y^j_{,b} \tag{4'b}$$

Again part of the equations (4'a) expresses charge conservation: the covariant intrinsic charge current density $\tilde{Q}^a = \sqrt{M}\,m^{ab}Q_b$ is conserved, while the contravariant effective charge current density $\tilde{I}^a = \sqrt{M}\,m^{ab}\,I_b$ is covariantly conserved:

$$\partial_a\,\tilde{I}^a + y^i_{,a}\,[A_i,\tilde{I}^a] = 0 \tag{5'}$$

The equations (4a) and (4'a) plus the boundary condition (BC) insure that the currents in space-time $J^i(x)$ are covariantly conserved.

Finally it should be stressed that, due to the parametrisation invariance of the theory, not all of the $4+N$ equations (3) or (3') are independent. For a point particle we have one, and for the string we have two identities:

$$\frac{\dot{z}^A}{\dot{\ell}}\;G_{AB}\left(\frac{1}{\dot{\ell}}\frac{d}{du}\frac{\dot{z}^B}{\dot{\ell}} + \Gamma^B_{MN}\frac{\dot{z}^M\,\dot{z}^N}{(\dot{\ell})^2}\right) \equiv 0 \tag{6}$$

$$z^A_{,c}\;G_{AB}\left(\frac{1}{\sqrt{M}}\,\partial_a(\sqrt{M}\,m^{ab}\,z^B_{,b}) + \Gamma^B_{MN}\,m^{ab}\,z^M_{,a}\,z^N_{,b}\right) \equiv 0 \tag{6'}$$

These are geometrical identities and cannot impose constraints on the metric.

In the abelian case, similar equations for motion are obtained with $\eta_{\alpha\beta}$ replaced by -1 and without the cosmological term in the Einstein equations.

4. DISCUSSION OF THE EQUATIONS OF MIOTION

We will restrict our discussion to the case of a fixed Minkowsky space-time metric.

Our aim is to solve equations (4a) and (4'a) describing the motion in the fibre, to substitute the solution in the equations (4b) and (4'b) and to obtain in this way a set of equations relating only the fields and the trajectories of matter in 4-dimensional space-time.

a) Point Particles

Equation (4a) is easy to solve: V/ℓ is a constant matrix in the Lie algebra of G and the intrinsic charge $Q(u)$ is constant along the worldline of the particle: $Q(u) = qK$. One obtains $d\ell^2 = ds^2 + a^2 \mathrm{Tr}(V^2) du^2 = ds^2 + \mathrm{Tr}(K^2) d\ell^2$ and, since $\mathrm{Tr}(K^2)$ is negative definite, it follows that $d\ell^2 > 0$ implies $ds^2 > 0$ and vice-versa. $d\ell/ds = (1 - \mathrm{Tr}(K^2))^{-1/2}$ is constant and the equations of motion will simplify if we use the arc--length s as parameter. From the definition of V it follows that

$$\dot{g} = - g\, \ell K/a - A_i\, \dot{y}^i\, g$$

which has the formal solution $g(s) = M(A,\gamma_s)\, g_0 \exp(-K\, \ell s/a)$ where

$$M(A,\gamma_s) = \left(\exp\left(- \int_{\gamma_s} A_i\, dy^i\right)\right)_+$$

is the path-ordered integral along γ_s, the trajectory of the point-particle in space-time from $y(0)$ to $y(s)$. Geometrically $M(A,\gamma_s)$ provides the isomorphism between the fibre over $y(0)$ and the fibre over $y(s)$, along the path γ_s. The effective charge is obtained as:

$$I(s) = M(A,\gamma_s)\, I_0\, M^{-1}(A,\gamma_s) \tag{7}$$

with $I_0 = q\, g_0\, K\, g_0^{-1}$.

This could also have been obtained directly from the covariant conservation law[5]

The Yang-Mills-Wong equations are now written as

$$\nabla_i F^{ij}(x) = 4\pi \sum_{\text{point}} \int ds\, M I_0 M^{-1} \dot{y}^j(s)\, \delta^4(x-y(s)) \qquad (8a)$$

and (*)

$$m' c^2 \ddot{y}^i(s) = \mathrm{Tr}(M I_0 M^{-1} F^i{}_j(y(s)))\, \dot{y}^j(s) \qquad (8b)$$

Under a gauge transformation, given in a local gauge by $g_\tau(x)$, one has:

$p \to p' = \tau(p)$ or $x \to x' = x$ and $h \to h' = g_\tau(x) h$

$\alpha(x) \to \alpha'(x) = g_\tau(x)\ \alpha(x)\ g_\tau^{-1}(x) - dg_\tau(x)\ g_\tau^{-1}(x)$

$\Phi(x) \to \Phi'(x) = g_\tau(x)\ \Phi(x)\ g_\tau^{-1}(x)$

$M(A,\gamma_s) \to M(A',\gamma_s) = g_\tau(y(s))\ M(A,\gamma_s)\ g_\tau^{-1}(y(0))$

$g_0 \to g_0' = g_\tau(y(0))\ g_0$

$I_0 \to I_0' = g_\tau(y(0))\ I_0\ g_\tau^{-1}(y(0))$

and the equations (8a) and (8b) are manifestly gauge covariant

Considering (8a) as the field equations for a given motion of the particles in space-time, we may always perform a gauge transformation such that for each point-particle:

$$g_\tau(x)\,\delta^4(x - y(s)) = g^{-1}(s)\,\delta^4(x - y(s)) \ .$$

The Yang-Mills equations become local

(*): $m' = m(1-\mathrm{Tr}(K^2))^{-1/2}$ is the dressed mass. It appears also in the expression of the energy-momentum tensor when written in s- parametrization.

$$\nabla_i F'^{ij}(x) = 4\pi \sum_{\text{point}} ds\, I'_0\, \dot{y}^j(s)\, \delta^4(x - y(s))$$

$$I'_0 = q\,K$$

The solutions $A'_i(x)$ have to satisfy the gauge conditions

$$[A'_i(x), K]\, \delta^4(x - y(s)) = 0$$

which means that the gauge potentials along the trajectories of the point-particles have to stay in the Cartan subalgebra containing K.

If one assumes with H.Arodź[10] that; in the case of only one particle, the potential remains parallel to K, the theory is in fact abelian and for a spherical symmetric field the well-known solution of M.Ikeda and Y.Miyashi is obtained[11].

If, in the same spirit, (8b) is interpreted as the equation of motion of a particle in a given field, the non-local features will disappear only if

$$[M(A, \gamma_s),\; F_{ij}(y(s))] = 0$$

It is easy to see that the commutability of $A_k(x)$ with $F_{ij}(x)$ and all its covariant derivatives is sufficient for this condition to hold. This means that $A_k(x)$ belongs to the center of the holonomy algebra at x (see H.Loos[12])). Again when $A_k(x)$ has a fixed direction in the Lie algebra, this condition is trivially satisfied.

This short discussion is meant to advocate the use of equations (8a) and (8b) which are equivalent to the equations of Wong, since they display explicitely these non-local effects. These are due to the covariant conservation of the effective charge which seems to be the crux in the difficulties in recent work on non-abelian classical radiation[13]. In particular the approximation scheme of W.Drechsler could start from the sole equation (8b) instead of from the set of equations (2) and (5).

b) String-Like Matter

Introducing the Lie algebra valued one-form on the worldsheet

$$V(\xi) = V_a(\xi)\, d\xi^a = g^{-1}(\xi)\, \tilde{\alpha}(\xi)\, g(\xi) + g^{-1}(\xi)\, dg(\xi) \tag{9a}$$

where

$$\tilde{\alpha}(\xi) = A_i(y(\xi))\, y^i{}_{,a}(\xi)\, d\xi^a = A_a(\xi)\, d\xi^a$$

$$dg(\xi) = g_{,a}(\xi)\, d\xi^a$$

and the Hodge $*$ operator:

$$^*V = \sqrt{M}\, a^{ab}\, V_b\, \varepsilon_{ac}\, d\xi^c \qquad (*)$$

equation (4'a) becomes

$$d\, {}^*V = 0 \tag{10a}$$

Let

$$W = g\, V\, g^{-1} = \tilde{\alpha} + dg\, g^{-1} \tag{9b}$$

be the one-form associated with the effective charge current, equation (5') becomes

$$d\, {}^*W + \tilde{\alpha} \wedge {}^*W + {}^*W \wedge \tilde{\alpha} = 0 \tag{10b}$$

Substituting expression (9) in (10) yields a second order differential equation for g:

$$d\, {}^*(dg\, g^{-1}) + d\, {}^*\tilde{\alpha} + \tilde{\alpha} \wedge {}^*dg\, g^{-1} + {}^*dg\, g^{-1} \wedge \tilde{\alpha} = 0 \tag{11a}$$

Choosing harmonic de Donder coordinates on the worldsheet

$$m_{00} + m_{11} = 0 \quad \text{and} \quad m_{01} = 0$$

or in light-like coordinates $\zeta^{\pm} = 1/\sqrt{2}\, (\xi^0 \pm \xi^1)$,

---

(*) ε_{ab} is the antisymmetric tensor capacity such that $\varepsilon_{01} = +1$.

$$(y_{,\pm})^2 + a^2\ \mathrm{Tr}(V_\pm^2) = 0$$

equation (11a) becornes:

$$\partial_-(g_{,+}\ g^{-1}) + \partial_+(g_{,-}\ g^{-1}) + [A_-,\ g_{,+}\ g^{-1}] + [A_+,\ g_{,-}\ g^{-1}]$$

$$+ \partial_- A_+ + \partial_+ A_- = 0 \qquad (11b)$$

This equation describes the interaction of a chiral field on the string interacting with the gauge field $\tilde{\alpha}(\xi)$.

It is gauge covariant under the transformation $t(\xi)$ E G:

$$g(\xi) \to g'(\xi) = t(\xi)\ g(\xi)$$

$$\tilde{\alpha}(\xi) \to \tilde{\alpha}'(\xi) = t(\xi)\tilde{\alpha}(\xi)t^{-1}(\xi) - dt(\xi)\ t^{-1}(\xi)$$

Instead of working with the one differential equation of second order (11b), it might be more advantageous to deal with a system of two first -order differential equations. The first one being (10b) and the second one is the integrability condition of (9b):

$$dW + \tilde{\alpha} \wedge W + W \wedge \tilde{\alpha} - W \wedge W = \tilde{\Phi}$$

where

$$\tilde{\Phi} = d\tilde{\alpha} + \tilde{\alpha} \wedge \tilde{\alpha} = 1/2\ F_{ab}(\xi)\ d\xi^a \wedge d\xi^b$$

$$F_{ab} = \partial_a A_b - \partial_b A_a + [A_a, A_b]$$

In harmonic light-like coordinates we obtain:

$$\partial_- W_+ + [A_-, W_+] = 1/2\ (F_{-+} + [W_-, W_+])$$

$$\partial_+ W_- + [A_+, W_-] = -1/2\ (F_{-+} + [W_-, W_+]) \qquad (12)$$

In the abelian case the Cauchy problem for equation (11b) or for the system (12) is trivial to solve.

Also in the case of a pure gauge ($F=0$) on the worldsheet, the potential can be gauged away and one encounters the equations of a principal chiral field in two dimensions:

$$\partial_- W_+ = 1/2 \, [W_-, W_+]$$

$$\partial_+ W_- = -1/2 \, [W_-, W_+]$$

or

$$\partial_-(g_{,+} g^{-1}) + \partial_+(g_{,-} g^{-1}) = 0$$

This system can be treated by the inverse scattering method[14].

5. OUTLOOK

It is known in the abelian case[15] that global properties of the $U(1)$ - bundle impose an integrality condition on the curvature, i.e. the electromagnetic field strength, which leads to the quantization of the charge. A similar condition should be found for non-abelian structure groups.

It should also be emphasized that the equations of motion as they stand, are to be renormalized. As mentioned in the introduction, the choice of a renormalization procedure based on a perturbative scheme using Riesz potentials[13] mostlikely excludes any confinement mechanism. In the abelian case, the only known bound state solution is the one of A. Schild[16] of the non-conventional action-at-a-distance electrodynamics of A.D.Fokker, P.A.M.Dirac and of J.A.Wheeler and Feynman[17].

Such a theory, a fortiori such a solution does not exists in the non-abelian case.

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