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# Quantum Non-Local Charge and Exact S-Matrix of the Gross-Neveu Model

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Using non perturbative methods we prove conservation of the non-local quantum charge of the Gross-Neveu model, providing an exact S-matrix.

Usando métodos não-perturbativos demonstra-se a conservação da carga quântica não local do modelo de Gross-Neveu, fornecendo uma matriz S exata.

## INTRODUCTION

The Gross-Neveu model[1] has been extensively studied in the last years. It is asymptotically free, displays mass transmutation, and has a well defined $1/N$ expansion[1]. It has been proved that the model has the factorization property in lowest order, providing a calculable $S$-matrix[2]. Afterwards, this S-matrix was extended to second order in $1/N$ perturbation theory[3]. However, we do not have, up to now, a general proof of the factorization property of this model outside the framework of perturbation theory, in contrast to the case of the non linear $O(n)$ symmetric a model[4]. In this model we use the same approach of ref. 4 to show that the factorization property is a non-perturbative feature of the model.

In section 1 we define the model and the classical non local charge. In section 2 we define the quantum non local. In section 3 we write it in terms of asymptotic fields. In section 4 we prove absence of particle production and the factorization equations. Section 5 is the conclusion.

## 1. THE MODEL AND THE NON LOCAL CHAIRGE

The Gross-Neveu model, is defined by the lagrangean density

$$L = \bar{\psi}_\alpha \gamma^\mu \partial_\mu \psi_\alpha + \frac{g^2}{2} (\bar{\psi}_\alpha \psi_\alpha)^2 \qquad (1.1)$$

and describes 2N Majorana fields in 1+1 space-time dimensions. We choose the following representation for the $\gamma$ matrices

$$\gamma^0 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \sigma_y \qquad (1.2a)$$

$$\gamma^1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} = i\,\sigma_x \qquad (1.2b)$$

$$\gamma^5 = \gamma^0\gamma^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad (1.2c)$$

$$\bar{\psi} = \psi^T \gamma^0$$

The Fourier decomposition $\psi_{in}$ reads (free case)

$$\psi^d_{in}(x) = \int_{-\infty}^{\infty} d\mu(p) \; [b^d(p) u(p) e^{-ip^0x^0 + ipx^1} +$$

$$+ b^d \; (p) u(p) e^{ip^0\gamma^0 - ipx^1}] \qquad (1.3)$$

$$d\mu(p) = dp\sqrt{m/2\pi p^0} \qquad (1.3a)$$

$$p^0 = \sqrt{m^2+p^2}$$

$$\{b^+_j(p),\; b_k(p')\} = \delta(p-p')\delta_{jk} \qquad (1.3b)$$

and for the $(m)$ two point Wightman function:

$$<0\left[\bar{\psi}^{a}_{in,\alpha}(0)\ \psi^{b^{+}}_{in,\beta}(y)\right]0> = \delta^{ab}\ S_{+}(y)_{\alpha\beta} =$$

$$= \delta^{ab}\left[\gamma^{0}(\not{p}+m)\right]_{\alpha\beta}\ \frac{1}{i}\ \Delta_{+}(y) \tag{1.4}$$

$$\frac{1}{i}\ \Delta_{+}(y) = \int \frac{dp}{2\pi p^{0}}\ e^{-ip^{0}y^{0}+ipy'} = \frac{1}{2\pi}\ K_{0}(m\sqrt{-y^{2}}) \tag{1.5}$$

The model has a conserved Noether current associated to the $O(2N)$ symmetry:

$$J^{ab}_{\mu}(x) = 2i\ \bar{\psi}^{a}_{(x)}\ \gamma_{\mu}\ \psi^{b}_{(x)} = -\ J^{ba}_{\mu}(x) \tag{1.6}$$

This current satisfies the so-called integrability condition[5]

$$\partial_{\mu}J^{ab}_{\nu} - \partial_{\nu}J^{ab}_{\mu} - 2g^{2}(J^{ac}_{\mu}J^{cb}_{\nu} - J^{ac}_{\nu}J^{cb}_{\mu}) = 0 \tag{1.7}$$

which allows us to write down the conserved non-local charge

$$Q^{ab} = \int_{-\infty}^{\infty} dy_{1}dy_{2}\ \varepsilon(y_{1}-y_{2})\ J^{ac}_{0}(t,y_{1})\ J^{cb}_{0}(t,y_{2})$$
$$-\ \frac{1}{g^{2}}\ \int_{-\infty}^{\infty} dy\ J^{ab}_{1}(t,y) \tag{1.8}$$

## 2. QUANTUM DEFINITION OF THE NON-LOCAL CHARGE

In field theory, the expression (1.8) is ill defined, due to the divergence of the product of two currents in the first integral, which displays a linear divergence for small $|y_{1}-y_{2}|$. We look for a Wilson expansion[6] for the product of two currents, which can be achieved and put in the form of a theorem:

Theorem: The Wilson expansion in the Gross-Neveu model for the product of two currents is given by

$$[J_\mu(z), J_\nu(0)]^{ab} = [C_1(z^2) z^2 g_{\mu\nu} z^\rho + C_2(z^2) z^2 (z_\mu \delta^\rho_\nu + z_\nu \delta^\rho_\mu) + C_3(z^2) z_\mu z_\nu z^\rho] J^{ab}_\rho(0) + [D_1(z^2) z^\rho (z_\mu \delta^\sigma_\nu - z_\nu \delta^\sigma_\mu) + D_2(z^2) z^\sigma (z_\mu \delta^\rho_\nu - z_\nu \delta^\rho_\mu) + \frac{1}{2} C_2(z^2) z^2 z^\sigma (z_\mu \delta^\rho_\nu + z_\nu \delta^\rho_\mu) + \frac{1}{2} C_3(z^2) z_\mu z_\nu z^\sigma z^\rho] \partial_\sigma J^{ab}_\rho(0) + O(|z|^{1-0}) \tag{2.1}$$

where

$$C_1(z^2) = -\frac{(n-2)}{\pi} \frac{1}{(z^2)^2} + O(|z|^{-2-0}) \tag{2.2a}$$

$$C_2(z^2) = \frac{(n-2)}{\pi} \frac{1}{(z^2)^2} + O(|z|^{-2-0}) \tag{2.2b}$$

$$C_3(z^2) = 0 \tag{2.2c}$$

$$D_1(z^2) = \frac{(n-2)}{4\pi} \frac{\ln(-z^2\mu^2)}{z^2} + O(|z|^{-1-0}) \tag{2.2d}$$

$$D_2(z^2) = -\frac{(n-2)}{4\pi} \frac{(\ln(-\mu^2 z^2)+2)}{z^2} \quad O(|z|^{-1-0}) \tag{2.2e}$$

and $\mu$ is related to the fermion mass by

$$\mu = \frac{1}{2} m e^\gamma, \; \gamma = 0.577\ldots \text{ is the Euler-Mascheroni constant.} \tag{2.3}$$

The proof of this theorem goes as follows. We first prove a lemma:

$$J^{ab}_\mu(z)\psi^c(0) = -\frac{1}{2\pi} \frac{\not z \gamma_\mu}{z^2} (\delta^{ac}\psi^b(0) - \delta^{bc}\psi^a(0)) + O(|z|^{-0}) \tag{2.4}$$

if the current is normalized by

$$\int dz^1 \; i\left[J_0^{ab}(z),\ \psi^c(0)\right]_{z^0=0} = \delta^{ac}\psi^b(0) - \delta^{bc}\psi^a(0) \qquad (2.5)$$

The proof is straightforward, using the fact that this current has to be proportional to $\gamma^\mu$ (remembering that $J_\mu^{ab} = 2i\,\bar\psi^a\gamma_\mu\,\psi^b$) and current conservation.

The most general Wilson expansion has the form:

$$\left[J_\mu(z), J_\nu(0)\right]^{ab} = C_{\mu\nu}^{\rho}(z)\; J_\rho^{ab}(0) + D_{\mu\nu}^{\sigma\rho}(z)\partial_\sigma J_\rho^{ab}(0)\ . \qquad (2.6)$$

Using locality, PT and CP invariance, we have the relations[4]:

$$C_{\mu\nu}^{\rho}(z) = -\,C_{\nu\mu}^{\rho}(-z) = C_{\nu\mu}^{\rho}(z) \qquad (2.7a)$$

$$D_{\mu\nu}^{\sigma\rho}(z) = -\,D_{\nu\mu}^{\sigma\rho}(-z) - \left|C_{\nu\mu}^{\rho}(-z)z^\sigma - \frac{1}{2}\,g^{\sigma\rho}C_{\nu\mu}^{\lambda}(-z)z_\lambda\right| \qquad (2.7b)$$

$$D_{\mu\nu}^{\sigma\rho}(-z) = D_{\mu\nu}^{\sigma\rho}(z) \qquad (2.7c)$$

and with Lorentz invariance, we have (2.1) as the most general Wilson expansion.

Current conservation implies:

$$\frac{d}{dz^2}\left(z^2 D_1\right) = -\,\frac{1}{4}\,C_1\;z^2 \qquad (2.8a)$$

$$\frac{d}{dz^2}\left(z^2 D_2\right) = -\,\frac{1}{4}\,C_2 z^2 \qquad (2.8b)$$

$$z^2\,\frac{d}{dz^2}\left(C_1+C_2+C_3\right) = -\left(C_1+C_2+2C_3\right) \qquad (2.8c)$$

$$z^2\,\frac{dC_2}{dz^2} = -\,\frac{1}{2}\left(C_1 + 5C_2\right) \qquad (2.8d)$$

$$D_1 + D_2 = \frac{1}{4}\left(C_1 + C_2 + C_3\right)\,z^2 \qquad (2.8e)$$

The above equations do not determine the coefficients completely. We use now:

$$i\left[\left[J_\mu(z),\ J_\nu(0)\right]^{ab},\ \psi^d(t,0)\right]_{z^0=0} =$$

$$= i\ C^\rho_{\mu\nu}(z)\Big|_{z_0=0}\ \left|J^{ab}_\rho(0),\ \psi^d(t,0)\right| + O(|z|^{-0}) \qquad (2.9)$$

For $\mu=0$, $\nu=1$ and using (2.4), we get after some calculation:

$$C_2(z^2) = \frac{(n-2)}{\pi}\frac{1}{(z^2)^2} + O(|z|^{-2-0}) \qquad (2.10)$$

It follows from (2.8d) that $C_1$ is given by (2.2a) and from (2.8c)

$$C_3 = \frac{\lambda}{(z^2)^2} + O(|z|^{-2-0}) \qquad (2.11)$$

The normalization of the current, (2.5), implies:

$$\int_{\infty}^{\infty} dz^1\ i\left\{\left[J_0(z),J_\nu(0)\right]^{ab} - \left[J_0(z),J_\nu(0)\right]^{ba}\right\}_{z^0=0} =$$

$$= -2(n-2)\ J^{ab}_\nu(0) \qquad (2.12)$$

and this relation forces $\lambda=0$, giving (2.2c).

Now, $D_1$, $D_2$ follow directly from the current conservation (2.8a) and (2.8b).

Equation (2.3) can be obtained by exactly the same procedure outlined in ref. 4.

We can define the cut-off non local charge:

$$Q^{ab} = \frac{1}{2n}\int_{|y_1-y_2|\geq\delta} dy_1 dy_2\ \varepsilon(y_1-y_2)\ \left[J_0(t,y_1),\ J_0(t,y_2)\right]^{ab}$$

$$- \frac{n-2}{2\pi}\ \ln(\mu\delta)\int_{-\infty}^{\infty} dy\ J^{ab}_1(t,y) \qquad (2.13)$$

which is readily seen to be finite and conserved in the limit $\delta \to 0$, by using the theorem of this section.

## 3. ASSYMPTOTIC CHARGE

Because of the conservation of the charge $Q^{ab}_\delta$,

$$Q^{ab} = \lim_{\delta \to 0} Q^{ab}_\delta \, , \tag{3.1}$$

we can write:

$$Q^{ab} = \lim_{t \to -\infty} Q^{ab}_{\text{in}}(t) = \lim_{t \to \infty} Q^{ab}_{\text{out}}(t) \; . \tag{3.2}$$

It is our purpose to write expressions for the limits in (3.2). We write:

$$Q^{ab}_{\delta\,\text{in(out)}} = \frac{1}{n} \left[ A^{ab}_{\delta\,\text{in(out)}}(t) + (n-2) B^{ab}_{\delta\,\text{in(out)}} \right] \tag{3.3}$$

$$A^{ab}_{\delta\,\text{in}} = \int_{|y_1 - y_2| \geq \delta} dy_1 dy_2 \; \varepsilon(y_1 - y_2) \; J^{ac}_{0\,\text{in}}(t, y_1) J^{cb}_{0\,\text{in}}(t, y_2) \tag{3.4}$$

$$B^{ab}_{\delta\,\text{in}} = \frac{1}{2} \int_{|y_1 - y_2| \geq \delta} dy_1 dy_2 \; \varepsilon(\eta_1 - \eta_2) : \{ S_+(0, y_1 - y_2) \, (\psi^+_{a\,\text{in}}(t, y_1) \psi^b_{\text{in}}(t, y_2)$$

$$- \psi^+_{b\,\text{in}}(t, y_1) \psi_{a\,\text{in}}(t, y_2)) + S_+(0, y_2 - y_1) \; \psi^{b^+}_{\text{in}}(t, y_2) \psi^a_{\text{in}}(t, y_1)$$

$$- \psi^{a^+}_{\text{in}}(t, y_2) \psi^b_{\text{in}}(t, y_1)) \} : - \frac{1}{\pi} \; \ell n(\mu\delta) \int_{-\infty}^{\infty} dy \; J^{ab}_{1\,\text{in}}(t, y) \tag{3.5}$$

and analogously for the out fields.

Using (1.3) up to (1.5), and taking the limits (3.2), we have, after a long calculation:

$$\lim_{\substack{t\to+\infty \\ (-)}} A_{\substack{\text{out} \\ (\text{in})}}(t) = \underset{(+)}{-} A_{\substack{\text{out} \\ (\text{in})}} \tag{3.6}$$

$$\lim_{t\to\pm\infty} B_{\substack{\text{out} \\ (\text{in})}}(t) = B_{\substack{\text{out} \\ (\text{in})}} \tag{3.7}$$

where

$$A_{\text{in}} = \int dp_1\, dp_2\, \varepsilon(p_1-p_2) :(b^{a^+}_{\text{in}}(p_1)\, b^c_{\text{in}}(p_1)$$

$$- b^{c^+}_{\text{in}}(p_1)\, b^a_{\text{in}}(p_1))(b^{c^+}_{\text{in}}(p_2) b^b_{\text{in}}(p_2) - b^{b^+}_{\text{in}}(p_2) b^c_{\text{in}}(p_2)): \tag{3.8}$$

$$B_{\text{in}} = \frac{1}{i\pi} \int_{-\infty}^{\infty} dp\, \ln \frac{p^0+p}{m} :(b^{a^+}_{\text{in}}(p) b^b_{\text{in}}(p) - b^{b^+}_{\text{in}}(p) b^a_{\text{in}}(p)): \tag{3.9}$$

The out fields have the same expression.

Summing up:

$$Q^{ab}_{\text{in}} = \frac{1}{n}\left[A^{ab}_{\text{in}} + (n-2)\, B^{ab}_{\text{in}}\right] \tag{3.10}$$

$$\text{Qout} = \frac{1}{n}\left[- A^{ab}_{\text{out}} + (n-2)\, B^{ab}_{\text{out}}\right] \tag{3.11}$$

## 4. ABSENCE OF PARTICLE PRODUCTION, AND FERMION-FERMION SCATTERING

Using the fact that

$$Q^{ab^+} = Q^{ab}$$

we will achieve many constraints applying $Q^{ab}_{\text{in}}$ on the right and $Q^{ab}_{\text{out}}$ on the left, in the amplitude

$$<\alpha \text{ out } |Q^{ab}| \beta \text{ in}>$$

Calling $\theta$ the rapidity defined by

$$\theta = \ell n \frac{p^0 + p}{m} \quad , \tag{4.1}$$

we have, using (2.8) up (2.11)

$$Q^{ab}|\theta_1 c_1 \dots \theta_R c_R \text{ in}> = |\theta_1 d_1 \dots \theta_R d_R \text{ in}> (M^{ab}_{\text{in}})_{d_1 - d_R c_1 - c_R} \tag{4.2a}$$

$$<\theta_1 c_1, \dots, \theta_\ell c_\ell \text{ out}|Q^{ab} = (M^{ab}_{\text{out}})_{c_1 - c_\ell d_1 - d_\ell} <\theta_1 d_1, \dots, \theta_\ell d_\ell \text{ out}) \tag{4.2b}$$

$$M^{ab}_{\substack{\text{in}\\ \text{out}}} = \mp \frac{1}{n} \sum_{K<j=1}^{\ell} (I_k^{ac} I_j^{cb} - I_k^{bc} I_j^{ca}) + \frac{n-2}{i\pi n} \sum \theta_k I_k^{ab} \tag{4.3}$$

$$(I_k^{ab})_{d_k c_k} = \delta^a_{d_k} \delta^b_{c_k} - \delta^b_{d_k} \delta^a_{c_k} \tag{4.4}$$

Now we need an expression (ansatz) for:

$$<\alpha \text{ out}|\beta \text{ in} > \quad .$$

To achieve it we first prove that we have only elastic scattering.

Consider the amplitude

$$\sum_{c=1}^{2N} <\theta'_1 c'_1, \dots, \theta'_{2\ell} c'_{2\ell} \text{ out}|Q^2| \theta_1 c, \theta_2 c \text{ in} > \tag{4.5}$$

$Q^2$ comutes with the isospin operator $J^{ab}$, so that the state $\sum_c |\theta_1 c, \theta_2 c \text{ in} >$ is an eigenstate of $Q^2$. Its eigenvalue can be easily calculated, using (3.10):

$$Q^2 \sum_n |\theta_1 c, \theta_2 c \text{ in}> = \lambda \sum_c |\theta_1 c, \vartheta_2 c \text{ in} > \tag{4.6}$$

$$\lambda = 2(n-1)\ \left(\frac{n-2}{n}\right)^2 \quad \left(1 + \frac{\theta^2}{\pi^2}\right) \tag{4.7}$$

$$\theta = \theta_1 - \theta_2$$

Now let us make $Q^2$ act on the left. We should remember that if the amplitude

$$<\theta_1' c_1', \ldots, \theta_{2\ell}' c_{2\ell}'\ \text{out}|\theta_1 c_1, \theta_2 c_2\ \text{in}> \tag{4.8}$$

is non zero for some value of the set $\theta_i'$, $\theta_i$, (4.5) should also be non zero, because $J^{ab}$ and $Q^2$ act irreducibly on states of particles with definite momenta.

The eigenvalue of $Q^2$, when acting on the left of (4.5) in zero rapidity ($\theta_1 = \theta_2 = \ldots = \theta_{2\ell} = 0$) is given by the value of (4.7) on the threshold:

$$\lambda = 2(n-1)\left(\frac{n-2}{n}\right)^2 \left(1+4\left|\frac{1}{\pi}\ \ell n(\ell + \sqrt{\ell^2-1})\right|^2\right) \tag{4.9}$$

However $\lambda$ should be an algebraic number, because $Q^2$ acting on the left is a matrix with rational coefficients and

$$a = \frac{1}{\pi}\ \ell n\ (\ell + \sqrt{\ell^2-1})$$

should be an algebraic number, which is impossible by a theoremofnumber theory[5, 7].

So we conclude:

$$<\theta_1' c_1', \ldots, \theta_{2\ell}' c_{2\ell}'\ \text{out}|\theta_1 c_1,\ \theta_2 c_2\ \text{in}> = 0 \tag{4.10}$$

and as a consequence:

$$<\theta_1' c_1', \theta_2' c_2'\ \text{out}|\theta_1 c_1, \theta_2 c_2\ \text{in}> =$$

$$= (4\pi)^2\ \delta(\theta_1'-\theta_1)\delta(\theta_2'-\theta_2)\{\delta^{c_1' c_2'}\ \delta^{c_1 c_2}\ \sigma_1(\theta)$$

$$+ \delta^{c_1' c_1}\delta^{c_2' c_2}\ \sigma_2(\theta) + \delta^{c_1' c_2}\ \delta^{c_2' c_1}\sigma_3(\theta)\} - (\theta_1 \leftrightarrow \theta_2) \tag{4.11}$$

The equation:

$$\langle\theta_1' c_1', \theta_2' c_2' \text{ out } (Q^{ab}_{\text{out}})^{+} \, |\theta_1 c_1, \theta_2 c_2 \text{ in}\rangle$$

$$= \langle\theta_1' c_1', \theta_2' c_2' \text{ out}| \, Q^{ab}_{\text{in}} |\theta_1 c_1, \theta_2 c_2 \text{ in}\rangle \tag{4.12}$$

gives a set of equations for a,, a,, o,, which can be solved, giving as result:

$$\sigma_1(\theta) = \frac{2\pi i}{i\pi - \theta} \frac{\sigma_2(\theta)}{n-2} \tag{4.13}$$

$$\sigma_3(\theta) = -\frac{2\pi i}{\theta} \frac{\sigma_2(\theta)}{n-2} \tag{4.14}$$

## 5. CONCLUSION

The Gross-Neveu model has the factorization property exactly. When we proved absence of particle production, it was already enough to prove this assertion because of a well known theorem on S-matrix theory[8]. To calculate the exact total S-matrix, we need to know the bound state spectrum. This is already used in the calculation of ref. 2, and what we proved is that their S-matrix is the exact one.

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