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Non-Holonomy and the Diffusion of Particles

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The diffusion equation for non-holonomic systems is derived and discussed. This equation, valid in the limit of high viscosíty, exhibits in the equilibrium state permanent solenoidal currents. These currents disappear when the system is Liouvillian, a concept introduced to generalize the behaviour of holonomic systems. It is further shown the continuity of the diffusion equation from non-holonomy to holonomy.

A equação de difusão para sistemas não-holônomos é obtida e discutida. Esta equação, válida no limite de alta viscosidade, exibe, no estado de equilíbrio, correntes solenoidais permanentes. Essas correntes desaparecem quando o sistema torna-se Liouviliano, um conceito introduzido para generalizar o comportamento dos sistemas holônomos. Mostra-se ainda que a equação de difusão apresenta continuidade quando se passa dos sistemas não-holônomos para os sistemas holônomos.

1. INTRODUCTION

Constraints are usually of the form

$$a_i(q)dq^i = 0$$

and represent a restriction on the possible displacements of the system. In the above equation q^i are the coordinates of the system and $a_i(q)$ are the components of a vector that points in the direction where the motion is forbiden.

Let us suppose the system moves in an n-dimensional manifold subject to m $(m<n)$ constraints*

$$\omega^\alpha \equiv a_i^\alpha dq^i = 0 \quad , \quad a = 1,\ldots,m \tag{1.1}$$

We write $d\omega^a$ for the exterior derivative of ω^α i.e.:

$$d\omega^\alpha = \frac{\partial a_i^\alpha}{\partial q^j} dq^j \wedge dq^i$$

where Λ is the notation for exterior product of forms. The system (1.1) is integrable if and only if there exists l-forms θ_β^α such that

$$\omega^\alpha = \sum_{\beta=1}^{m} \theta_\beta^\alpha \wedge \omega^\beta \quad ; \tag{1.2}$$

otherwíse the system is not integrable. Mechanical systems subject to non-integrable constraints are called non-holonomic systems.

If the system of eqs. (1.1) is integrable then one can find m functions $\phi^\alpha(q)$ representing hypersurfaces in the n-dimensional manifold of configurations of the system. The motion is therefore restricted to one of the submanifolds defined by the intersection of these m-hypersurfacesandcan be thought as an unconstrained motion in this $(n-m)$ dimensional configuration space. As far as the motion of the system is concerned, the consequences of restricting it to a subrnanifold is, besides the reduction in the degrees of freedom, the modification of the metric of the configuration space. We will assume from now on that the rnetric is given by an arbitrary fundamental tensor g_{ij} and therefore the kinetic energy T of the system has the general form

$$T = \frac{1}{2} g_{ij} \dot{q}^i \dot{q}^j \tag{1.3}$$

* Repeated latin indices subintends summation.

The fact that the system (1.1) is not integrable does not exclude the possibility of having integrable subset of constraint equations. If this is so, we may always assume, without loss of generality, that these subsets of equations have been integrated and the configuration space reduced correspondingly. Thus, it is no fundamental restriction to assume that for the system of constraints under considerations every subset of equations is also not integrable.

The equations of motion under the influence of a generalized force F_i are:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}^i}\right) - \frac{\partial T}{\partial q^i} = \sum_{\alpha=1}^{m} \lambda_\alpha a_i^\alpha + F_i \tag{1.4}$$

These are D'Alembert's equation and λ_a are Lagrange multipliers that can be elliminated by the use of eqs. (1.1).

Thus, from eq. (1.4) we get

$$\ddot{q}^i + \Gamma^i_{jk}\,\dot{q}^j\dot{q}^k = \sum_{\alpha=1}^{m} \lambda^\alpha a^i_\alpha + F^i \tag{1.5}$$

where

$$\Gamma^i_{jk} = \frac{1}{2}\, g^{i\ell}\left(\frac{\partial g_{\ell j}}{\partial q^k} + \frac{\partial g_{k\ell}}{\partial q^j} - \frac{\partial g_{jk}}{\partial q^\ell}\right) \tag{1.6}$$

is the affine connexion associated to the metric tensor g^{ij}.

We may assume, without loss of generality, that

$$a_i^\alpha\, a^i_{\alpha'} = \delta^\alpha_{\alpha'}$$

where

$$a^i_{\alpha'} = g^{ij}\, a_j^{\alpha'} \quad .$$

From

$$a_i^\alpha\, \dot{q}^i = 0 \quad , \qquad \alpha = 1,\dots,m \tag{1.6a}$$

we obtain

$$\frac{\partial a_i^\alpha}{\partial q^j} \dot{q}^i \dot{q}^j + a_i^\alpha \ddot{q}^i = 0 \quad . \quad \alpha = 1,\ldots,m \tag{1.7}$$

Using eq. (1.5) into eq. (1.7) we obtain

$$\lambda^\alpha = - a^\alpha_{j;k} \dot{q}^j \dot{q}^k - a^\alpha_j F^j \tag{1.8}$$

where we introduced the covariant derivative notation

$$a^\alpha_{j;k} = \frac{\partial a^\alpha_j}{\partial q^k} - \Gamma^i_{jk} a^\alpha_i$$

Elliminating λ^α from eq. (1.7) we finally arrive at

$$\ddot{q}^i + \Gamma^i_{jk} \dot{q}^j \dot{q}^k = - \sum_{\alpha=1}^{m} a^i_\alpha a^\alpha_{j;k} \dot{q}^j \dot{q}^k + Q^i_j F^j \tag{1.9}$$

where Q^i_j is the following projector operator

$$Q^i_j = \delta^i_j - \sum_{\alpha=1}^{m} a^i_\alpha a^\alpha_j \tag{1.10}$$

We consider eq. (1.9) together with eqs. (1.1) the fundamental equations of motion for non-holonomic systems.

Let at this point observe that if eqs. (1.9) were equivalent to a hamiltonian system, the problem of obtaining the diffusion equation for non-holonomic systems would be greatly simplified as Hamilton's equations are manifestly covariant and Liouville's theorem ensures the existence of an invariant measure of probability in the phase space of the system.

Let us for the moment set the applied forces equal to zero ($F^j=0$ in eq. (1.9)). A hamiltonian associated to the constrained system is

$$H = \frac{1}{2} Q^{ij} p_i p_j \tag{1.11}$$

where Q^{ij} is given by equation (1.10). The equations of motion for the system described by H are

$$\dot{q}^i = \frac{\partial H}{\partial p_i} = Q^{ij} p_j$$
$$\dot{p}^i = -\frac{\partial H}{\partial q^i} = \sum_{\alpha=1}^{m} \frac{\partial a^j_\alpha}{\partial q^i} p_j a^k_\alpha p_k \tag{1.12}$$

and we observe that from the first set of Hamilton's equations we have

$$a^\alpha_i \dot{q}^i = 0$$

what shows that the above system obeys the constraints given by eqs. (1.1).

From eqs. (1.12) we obtain

$$\ddot{q}^i + \Gamma^i_{jk}\dot{q}^j\dot{q}^k = -\sum_{\alpha=1}^{m} a^i_\alpha a^\alpha_{j;k}\dot{q}^j\dot{q}^k + Q^{ij}\sum_{\alpha=1}^{m} P_\alpha \left(\frac{\partial a^\alpha_j}{\partial q^k} - \frac{\partial a^\alpha_k}{\partial q^j}\right)\dot{q}^k \tag{1.13}$$

where we have set

$$P_a = a^i_a p_i \ ,$$

Let us observe that P_α are the components of the momentum in the forbiden directions of motion and as such they are not directly connected to the velocities of the system, these being given by $Q^{ij}p_i$, the components of p_j in the allowed directions of motion. Therefore P_α are not fixed by the initial condition of the trajectory. They behave rather as control variables to lead the motion from an initial position to a pre-established final point in the configuration space at a fixed time interval. For non-holonomic systems there is always a possible trajectory between two fixed points in the n-dimensional configuration spaceeven though the system has only $(n-m)$ degreesof freedom. In this case, P_α plays the role of additional parameters to compensate the lackof degrees of freedom.

In order that eq. (1.13) should agree with eq. (1.9) it is necessary and sufficient that the relation

$$Q^{ij}\sum_{\alpha=1}^{m} P_\alpha \left(\frac{\partial a^\alpha_j}{\partial q^k} - \frac{\partial a^\alpha_k}{\partial q^j}\right)\dot{q}^k = 0$$

be valid for every possible values of P_α snd $\dot{q}^k$. This is equivalent to assume that there exists $\theta^\alpha_{\alpha' k}$ such that

$$\left(\frac{\partial a^\alpha_j}{\partial q^k} - \frac{\partial a^\alpha_k}{\partial q^j}\right) dq^k = \sum_{\alpha'=1}^{m} \theta^\alpha_{\alpha' k}\, a^{\alpha'}_j\, dq^k$$

Because of eq. (1.6a) we also have

$$\left(\frac{\partial a^\alpha_j}{\partial q^k} - \frac{\partial a^\alpha_k}{\partial q^j}\right) dq^k = \sum_{\alpha'=1}^{m} \theta^\alpha_{\alpha' k}\, a^{\alpha'}_j\, dq^k - \sum_{\alpha'=1}^{m} \theta^\alpha_{\alpha' j}\, a^{\alpha'}_k\, dq^k$$

which, in exterior-form notation, can be written as

$$d\omega^\alpha = \sum_{\alpha'=1}^{m} \theta^\alpha_{\alpha'} \wedge \omega^{\alpha'}$$

with

$$\theta^\alpha_{\alpha'} = \theta^\alpha_{\alpha' k}\, dq^k \quad .$$

This shows that the dynamics described by H given by eq. (1.11) agrees with the dynamics of constrained systems if and only if the constraints are integrable. In spite of this fact, the Hamiltonian given by eq. (1.11) was used in reference 1 to obtain a generalization of the diffusion equation for non-holonomic systems in the hope that this generalization would contain the holonomic case as a limiting case. This was observed not to be so and actually the generalized diffusion.equation there obtained predicted, in the equilibrium state a uniform density distribution over the whole configuration space, independent of the nature of the constraint, and contrary to the holonomic limit[1].

In this paper we will develop the theory of diffusion for non--holonomic systems, directly from eqs. (1.9) with the help of Fokker--Planck equation for Markovian systems. We will generalize Fokker-Planck equation for an arbitrary measure of probability in phase space and we will choose this measure in such a way to ensure covariances of the equation with respect to arbitrary point trarisformations. After establishing this fact we will deduce the hierarchy of hydrodynamical equations and we will show that the diffusion equation is the limit, near equilibrium, of an expansion in the mean free path of the particles. From the diffusion equation obtained we will prove that it goes into the correct limit ior holonomic systems. It reproduces also the result previously obtained[2] when the constraints, though non-holonomic, are such as to preserve the validity of Liouville's theorem of incompressible flow in phase space. We will further show that in general, non-holonomic systems may present, besides a non-uniform density, a permanent solenoidal current of particles in the equilibrium state.

2. THE FOKKER-PLANCK EQUATION

Let us consider $\omega(x_1,x_2;t)$ the transition probability for a Markovian process, i.e., the probability of finding a particle at the position x_2 of its N-dimensional phase space knowing that the particle was at x_2, t seconds earlier. The Chapman-Kolmogorov equation for this process is

$$\omega(x_1,x_2;t+t') = \int\mu(x_3)\,\omega(x_1,x_3;t)\,\omega(x_3,x_2;t')\,d^Nx_3 \tag{2.1}$$

where we have assumed that the phase space probability has a measure given by $\mu(x)$.

It is well known[3] that if we assume the following properties for $\omega(x_1,x_2;t)$;

(i) $$\langle\Delta x^i\rangle \equiv \int d^Nx_1\mu(x_1)\,(x_1^i-x_2^i)\,\omega(x_1,x_2;\Delta t) = A^i(x_2)\Delta t + 0(\Delta t^2) \tag{2.2}$$

(ii) $$\langle\Delta x^i\Delta x^j\rangle \equiv \int d^Nx_1\mu(x_1)\,(x_1^i-x_2^i)\,(x_1^j-x_2^j)\,\omega(x_1,x_2;\Delta t) = B^{ij}(x_2)\mathrm{At} + 0(\Delta t^2) \tag{2.3}$$

(iii) higher momenta of $\omega(x_1,x_2;\Delta t)$ are of order equal to or larger than Δt^2;

then, the following approximation of the Chapman- Kolmogorov equation, known as the Fokker-Planck equation, is obtained:

$$\frac{\partial \omega}{\partial t}(x_1,x_2;t) = -\frac{1}{\mu(x_1)}\frac{\partial}{\partial x_1^i}\left(A^i(x_1)\mu(x_1)\omega(x_1,x_2;t)\right) + \\ + \frac{1}{2\mu(x_1)}\frac{\partial^2}{\partial x_1^i \partial x_1^j}\left(B^{ij}(x_1)\mu(x_1)\omega(x_1,x_2;t)\right) \quad . \tag{2.4}$$

Let us define the one-particle distribution function $W(x;t)$ as

$$W(x;t) = \int d^N y \;\; \mu(y)\omega(x,y;t)W(y;0) \;\; ;$$

then, $W(x;t)$ satisfies the same differential equation as $\omega(x_1,x_2;t)$ and we have

$$\frac{\partial W}{\partial t}(x;t) = \frac{1}{\mu(x)}\frac{\partial}{\partial x^i}\left[A^i(x)W(x;t)\right] + \frac{1}{2}\frac{1}{\mu(x)}\frac{\partial^2}{\partial x^i \partial x^j}\left[B^{ij}(x)W(x;t)\right] \tag{2.5}$$

We apply this equation to the motion of non-holonomic systems described by eq. (1.9) when subject to a viscous force $-\zeta\dot{q}^j$ and a Langevin force $L^j(t)$. We assume ζ, the coefficient of viscosity, a constant scalar and to $L^j(t)$ we give the following statistical properties:

(i) $\quad <\int_0^{\Delta t} L^j(t)dt> = 0$

(ii) $\quad <\int_0^{\Delta t} dt' \int_0^{\Delta t} dt \; L^j(t')L^i(t)> = 2K\, g^{ij}\Delta t + O(\Delta t^2)$

(iii) higher order correlations of $L^j(t)$ are of order higher than or equal to Δt^2.

Under these conditions, besides being Markovian, the transition probability for the system satisfy the properties necessary for the validity of Fokker-Planck equation and we have:

$$<\Delta q^i> = v^i \Delta t$$

$$<\Delta v^i> = \left[-\left(\Gamma^i_{jk} + \sum_{\alpha=1}^{m} a^i_{\alpha} a^{\alpha}_{j;k}\right) v^j v^k - \zeta v^i\right] \Delta t \tag{2.7}$$

$$<\Delta v^i \Delta v^j> = 2K\, Q^{ij}\, \Delta t$$

as the only moments of the transition probability of first order in Δt irrespective of the choice of $\mu(x)$.

Let us introduce the vectors $b^{\beta}_{\cdot i}$ $(\beta=1,\ldots,n-m)$ such that

$$\text{(i)}\quad b^{\beta}_{\cdot i} a^i_{\alpha} = 0$$

$$\text{(ii)}\quad b^{\beta}_{\cdot i} b^i_{\beta'} = \delta^{\beta}_{\beta'}$$

where

$$b^i_{\beta'} = g^{ij}\, b^{\beta'}_{\cdot j}$$

We define the pseudo-velocities

$$u^{\beta} = b^{\beta}_{\cdot i}\, v^i$$

and observe that the phase space of the system is the direct sum of configuration nianifold and the space spanned by the pseudo-velocities. We further observe that

$$d^n q^i d^{n-m} u^{\beta} = d^n q^i d^n v^i \prod_{\alpha=1}^{m} \delta(a^{\alpha}_{\cdot i} v^i)$$

We will set

$$\mu = g(q) \prod_{\alpha=1}^{m} \delta(a^{\alpha}_{\cdot i} v^i) \tag{2.8}$$

as the measure of probability with

$$g(q) = \det\,(g_{ij}) \,,$$

and we take the phase space as the tangent fiber-bundle of the configuration manifold.

Making use of eqs. (2.7) and (2.8) into eq. (2.5) we obtain

$$\frac{\partial W}{\partial t} + \frac{v^i}{g}\frac{\partial}{\partial q^i}(gW) - \frac{\partial}{\partial v^i}\left[(\Gamma^i_{jk} + \sum_{\alpha=1}^{m} a^i_\alpha a^\alpha_{j;k})v^i v^k W\right] =$$

$$= \frac{\partial}{\partial v^i}\left[\zeta v^i W + KQ^{ij}\frac{\partial W}{\partial v^j}\right] \quad . \qquad (2.9)$$

We take the above equation as the fundamental equation to describe the stochastic motion of non-holonomic systems under the action of viscous and Langevin forces.

From now on we conclude this section proving that eq. (2.9) is covariant under the following general transformation:

$$\bar{q}^i = \bar{q}^i(q)$$
$$\bar{v}^i = \frac{\partial \bar{q}^i}{\partial q^j} v^j \qquad (2.10)$$

We first observe that using the following property of g:

$$\frac{1}{g}\frac{\partial g}{\partial q^i} = 2\Gamma^j_{ij} \qquad (2.10a)$$

we transform eq. (2.9) into

$$\frac{\partial W}{\partial t} + v^i\left[\frac{\partial W}{\partial q^i} - \Gamma^j_{ik}\frac{\partial W}{\partial v^j}v^k\right] =$$

$$= \frac{\partial}{\partial v^i}\left[\zeta v^i W + \sum_{\alpha=1}^{m} a^i_\alpha a^\alpha_{j;k} v^j v^k W + Q^{ij}\frac{\partial W}{\partial v^j}\right] \qquad (2.11)$$

The second member of this equation is manifestly invariant being the dívergence of a vector in the velocity subspace. This is so because the transformation (2.10) is independent of v^i.

We will now show that the following object

$$\frac{\partial W}{\partial q^i} - \Gamma^j_{ik} v^k \frac{\partial W}{\partial v^j}$$

behaves as a vector exhibiting therefore the invariant nature of eq. (2.9).

We have

$$W(q,v;t) = \bar{W}\left(\bar{q}^{i}(q),\ \frac{\partial \bar{q}^{i}}{\partial q^{j}}\, v^{j};t\right)$$

from the scalar nature of W

Therefore

$$\frac{\partial W}{\partial q^{i}} = \frac{\partial \bar{W}}{\partial \bar{q}^{j}} \frac{\partial \bar{q}^{j}}{\partial q^{i}} + \frac{\partial \bar{W}}{\partial \bar{v}^{j}} \frac{\partial^{2} \bar{q}^{j}}{\partial q^{i} \partial q^{k}}\ v^{k} \tag{2.12}$$

On the other hand we have

$$\frac{\partial^{2} \bar{q}^{j}}{\partial q^{i} \partial q^{k}} = \bar{\Gamma}^{\ell}_{ik} \frac{\partial \bar{q}^{j}}{\partial q^{\ell}} - \bar{\Gamma}^{j}_{mn} \frac{\partial \bar{q}^{m}}{\partial q^{k}} \frac{\partial \bar{q}^{n}}{\partial q^{i}}$$

Making use of the above relation into eq. (2.12) we finally obtain

$$\frac{\partial W}{\partial q^{i}} - \Gamma^{j}_{ik}\ v^{k} \frac{\partial W}{\partial v^{j}} = \left(\frac{\partial W}{\partial \bar{q}^{\ell}} - \bar{\Gamma}^{j}_{\ell k}\ \bar{v}^{k} \frac{\partial \bar{W}}{\partial \bar{v}^{j}}\right) \frac{\partial \bar{q}^{\ell}}{\partial q^{i}}$$

what exhibits the vector character of

$$\frac{\partial W}{\partial q^{i}} - \Gamma^{i}_{ik}\ v^{k} \frac{\partial W}{\partial v^{j}}$$

and therefore the invariant nature of Fokker-Planck equation.

3. THE HYDRODYNAMICAL EQUATIONS

We may look at eq. (2.8) as describing the motion of a fluid in the configuration manifold. The density of matter of this fluid is defined by

$$\rho(q;t) = \int \sqrt{g} \prod_{\alpha=1}^{m} \delta(a^{\alpha}_{i} v^{i}) W(q,v;t) d^{N}v \tag{3.1}$$

and its current as

$$j^{i} = \int \sqrt{g} \prod_{\alpha=1}^{m} \delta(a^{\alpha}_{k} v^{k}) v^{i} W(q,v;t) d^{N}v \tag{3.2}$$

These objects are particular cases of the definition of the ν-th moment of W in the velocity space:

$$P^{\ell_1;\ldots,\ell_\nu} = \int \sqrt{g} \prod_{\alpha=1}^{m} \delta(a^{\alpha}_{k} v^{k}) v^{\ell_1} \ldots v^{\ell_\nu} W(q,v;t) d^{N}v \tag{3.3}$$

from where we recognize ρ and j^{i} as the zeroth and first moments respectively.

We will assume that W vanishes sufficiently fast for $v_i v^i \to \infty$ in order that $P^{\ell_1,\ldots,\ell_\nu}$ to be defined by the integral in eq. (3.3). We further observe that $P^{\ell_1,\ldots,\ell_\nu}$ is a symmetric tensor in all its indices and

$$a^{\alpha}_{i_1} P^{i_1,\ldots,i_\nu} = 0 \ , \quad a = 1,\ldots,m$$

Integrating over the velocity space on both sides of eq. (2.9), we obtain the zeroth moment equation:

$$\frac{\partial \rho}{\partial t} + \frac{1}{\sqrt{g}} \frac{\partial}{\partial q^{i}} (\sqrt{g}\, j^{i}) = 0 \tag{3.4}$$

which expresses the conservation of matter

Multiplying both sides of eq. *(2.9)* by v^{i} and integrating over the velocity space we obtain the first moment equation:

$$\frac{\partial j^{i}}{\partial t} + \frac{\partial P^{ij}}{\partial q^{j}} + \Gamma^{k}_{kj} P^{ij} + \Gamma^{i}_{jk} P^{jk} + \sum_{\alpha=1}^{m} a^{i}_{\alpha} a^{\alpha}_{j;k} P^{jk} = -\zeta j^{i} \tag{3.5}$$

where we made use of eq. (2.10a). Similarly we could obtain the ν-th moment equation and eq. (2.9) results to be equivalent to a hierarchy of hydrodynamical equations, the equations for the moments of W in velocity space.

Before proceeding to write down the general form of these equations, let us introduce a new affine connexion defined by

$$\hat{\Gamma}^{i}_{jk} = \Gamma^{i}_{jk} + \sum_{\alpha=1}^{m} a^{i}_{\alpha} a^{\alpha}_{j;k} \tag{3.6}$$

and use the **colon** for the covariant derivative associated to this newconexion; for example

$$P^{ij}_{\;\;:k} = \frac{\partial P^{ij}}{\partial q^{k}} + \hat{\Gamma}^{i}_{k\ell} P^{\ell j} + \hat{\Gamma}^{j}_{k\ell} P^{i\ell}$$

and similarly, in the usual **fashion,** for every other tensor. With this notation, eq. (3.5) takes the form

$$\frac{\partial j^{i}}{\partial t} + P^{ij}_{\;\;:j} = -\zeta j^{i} \tag{3.7}$$

Because of

$$j^{i}_{\;:i} = j^{i}_{\;;i} = \frac{1}{\sqrt{g}} \frac{\partial}{\partial q^{i}} (\sqrt{g} \;\; j^{i})$$

we also have

$$\frac{\partial \rho}{\partial} + j^{i}_{\;:i} = 0 \tag{3.8}$$

for the zeroth moment equation. The ν-th moment equation takes the form (ν a 2)

$$\frac{\partial P^{\ell_1,\ldots,\ell_\nu}}{\partial t} + P^{\ell_1,\ldots,\ell_{\nu+1}}_{\qquad :\ell_{\nu+1}} = -\nu\zeta P^{\ell_1,\ldots,\ell_\nu} + 2K\left[Q^{\ell_1\ell_2} P^{\ell_3,\ldots,\ell_\nu}\right]_+ \tag{3.9}$$

where the symbol $[\;\;]_+$ stands for the symmetric combination in the indices of the direct product of the tensors inside the bracket. For example, we have

$$\frac{\partial P^{ij}}{\partial t} + P^{ijk}_{\;\;\;:k} = 2\, P^{ij} + 2KQ^{ij} \rho, \tag{3.10}$$

$$\frac{\partial P^{ijk}}{\partial t} + P^{ijk\ell}_{\;\;\;\;:\ell} = -3\zeta P^{ijk} + 2K(Q^{ij}P^{k} + Q^{ik}P^{j} + Q^{jk}P^{i}) \;, \tag{3.11}$$

$$\frac{\partial P^{ijk\ell}}{\partial t} + \dot{P}^{ijk\ell m}{}_{:m} = -4\zeta P^{ijk\ell} + 2K(\dot{Q}^{ij}P^{k\ell} + Q^{ik}P^{j\ell} + $$

$$+ Q^{i\ell}P^{jk} + Q^{jk}P^{i\ell} + \dot{Q}^{j\ell}P^{ik} + Q^{k\ell}P^{ij}) \tag{3.12}$$

and so on.

The fact that we can write the hydrodynamical equations in a new covariant notation using the asymmetric connexion given by eq. (3.6) can be drawn back to the fact that there exists a space, even for non-holonomic systems, to which the motion is referred as unconstrained. E. Cartan[4] was the first to call attention to this fact, and to exhibit the torsion of this space.

With the connexion defined by eq. (3.8), the equations of motion (1.9) can be written as

$$\frac{Dv^i}{Dt} = Q^i_{\;j}F^j$$

where

$$\frac{Dv^i}{Dt} = \frac{dv^i}{dt} + \hat{\Gamma}^i_{jk}\, v^j v^k$$

is the absolute acceleration in the space with the connexion $\hat{\Gamma}^i_{jk}$.

If the system is holonomic, then, we can find $\hat{g}_{ij}$ such that $\hat{\Gamma}^i_{jk}$ keeps the same relation to $\hat{g}_{ij}$ as expressed in eq. (1.6).

If the system is non-holonomic there is no such $\hat{g}_{ij}$ and besides the previous equation we also must keep the relations

$$a^\alpha_{.i} v^i = 0 \qquad , \qquad \alpha = 1,\dots,m$$

to complete the description of these systems.

4. THE DIFFUSION EQUATIONS

From now on we will proceed by looking for solutions near the steady state equilibrium of the fluid and we will neglect all partial derivatives with respect to time of all the moments but p, i.e.,

$$\frac{\partial P^{\ell_1,\ldots,\ell_\nu}}{\partial t} = 0 , \quad \nu \geq 1 . \tag{4.1}$$

Under this hypothesis we can rewrite the moment equations of the previous section as $(\nu \geq 2)$

$$P^{\ell_1,\ldots,\ell_\nu} = \frac{2K}{\nu\zeta}\left[Q^{\ell_1\ell_2}P^{\ell_3,\ldots,\ell_\nu}\right]_+ - \frac{1}{\nu\zeta}P^{\ell_1,\ldots,\ell_{\nu+1}}{}_{:\ell_{\nu+1}} \tag{4.2}$$

from where we observe that the v-th moment can be resolved into a (v - 2) -th moment and the divergence of the $(\nu+1)$-th moment.

Making sistematic use of the moment equations we can therefore reduce formally the hierarchy of equations into a unique equation for p. This is the generalized diffusion equation for the fluid.

Let us illustrate this procedure. First we have

$$j^i = -\frac{1}{\zeta} P^{ij}{}_{:j} \tag{4.3}$$

and

$$P^{ij} = \frac{K}{\zeta} Q^{ij}\rho - \frac{1}{2\zeta} P^{ijk}{}_{:k} \tag{4.4}$$

Therefore

$$\frac{\partial\rho}{\partial t} = \frac{K}{\zeta^2}\left(Q^{ij}\rho\right)_{:j:i} - \frac{1}{2\zeta^2} P^{ijk}{}_{:k:j:i} \tag{4.5}$$

Elliminating P^{ijk} in the above equation we would get terms whose sum of the order of derivatives would be equal to or larger than fourth and so on.

Let us define the diffusion coefficient as

$$D = \frac{K}{\zeta^2} \tag{4.6}$$

and the mean free path Λ:

$$\Lambda = \left(\frac{D}{\zeta}\right)^{1/2} \tag{4.7}$$

The equation for ρ can be formally written as

$$\frac{\partial\rho}{\partial t} = D\left(\sum_{\nu=0}^{\infty} \Lambda^{2\nu} K_\nu(Q)\rho\right) \tag{4.8}$$

where K_ν is an operator, functional of Q^{ij}, acting on ρ. $K_\nu\rho$ has the property of all its terms being homogeneous of degrees $(2\nu+2)$ in the sum of the order of derivatives of Q and ρ.

The first two terms of the series are:

$$K_0(Q)\rho = (Q^{ij}\rho)_{:j:i} \tag{4.9}$$

$$K_1(Q) = \frac{2}{3}\left\{\left[Q^{ij}(Q^{k\ell}\rho)_{:\ell}\right]_+ + \frac{1}{2}\left[Q^{ij}Q^{k\ell}\rho\right]_{+\ :\ell}\right\}_{:k:j:i} . \tag{4.10}$$

If we assume that $a^\alpha_i(q)$ varies slowly over a distance of the mean free path of the particle, i.e.,

$$\Lambda \frac{\partial a^\alpha_j}{\partial q^i} << 1 \quad , \quad \forall\alpha,\ i \text{ and } j\ ,$$

then one would expect that ρ would do the same. Under these hypothesis one is justified in keeping only the first term in $\Lambda^{2\nu}$ expansion of eq. (4.8) and we set

$$\frac{\partial\rho}{\partial t} = D(Q^{ij}\rho)_{:j:i} \tag{4.11}$$

as the diffusion equation for non-holonomic systems.

Let usobserve that if we set $Q^{ij} = g^{ij}$ in the above equation and take the usual covariant derivative we obtain the diffusion equation for free particles in a Riemannian manifold of metric g^{ij}.

5. DISCUSSION

We will discuss now some of the properties of eq. (4.11). To do so we begin by going back to the usual covariant derivative and we have

$$(Q^{ij}\rho)_{:j} = Q^{ij}\frac{\partial\rho}{\partial q^j} + \rho Q^{ij}{}_{:j} \quad (5.1)$$

From the definition given by eq. (3.6) we also have

$$Q^{ij}{}_{:j} = -\sum_{\beta=1}^{m} a^j_\beta a^i_{\beta;j} + \sum_{\alpha=1}^{m} a^j_\alpha a^i_\alpha \sum_{\beta=1}^{m} a^k_\beta a^\beta_{j;k} \quad (5.2)$$

where we made of the fact that

$$a^j_\alpha a^\alpha_{j;k} = 0$$

Because of

$$a^j_\alpha a^\beta_j = \delta^\beta_\alpha$$

we get

$$a^k_{\alpha;j} a^\beta_k = -a^k_{\beta;j} a^\alpha_k \quad (5.3)$$

Making use of the above equation into eq. (5.2) we arrive at

$$Q^{ij}{}_{:j} = -Q^{ij}\sum_{\beta=1}^{m} a^k_\beta a^\beta_{j;k} \quad (5.4)$$

and we may finally write

$$\frac{\partial\rho}{\partial t} = \frac{D}{\sqrt{g}}\frac{\partial}{\partial q^i}\left[\sqrt{g}\, Q^{ij}\left(\frac{\partial\rho}{\partial q^j} - \rho\sum_{\beta=1}^{m} a^k_\beta a^\beta_{j;k}\right)\right] \quad (5.5)$$

as the diffusion equation in the standard covariant notation.

We now discuss the holonomic limit.

In this case we may assume without loss of generality that

$$a_i = \frac{\dfrac{\partial \phi^\beta}{\partial q^i}}{A_\beta} \tag{5.6}$$

where

$$A_\beta(q) = \left(\frac{\partial \phi^\beta}{\partial q^i} \frac{\partial \phi^\beta}{\partial q^j} g^{ij} \right)^{1/2}$$

with the functions ϕ^β defining the integral of the system of constraints, i.e.,

$$\phi^\beta = \text{const.} \quad , \quad \beta = 1,\ldots,m \tag{5.7}$$

defining the leaves of the foliation.

From eq. (5.6) we obtain

$$a^k_\beta a^\beta_{j;k} = (\delta^\ell_j - a^\ell_\beta a^\beta_j) \frac{\partial}{\partial q^\ell} \log A_\beta$$

and we therefore get

$$Q^{ij} \sum_{\beta=1}^{m} a^k_\beta a^\beta_{j;k} = Q^{ij} \frac{\partial}{\partial q^j} \log \mu_0$$

where

$$\mu_0 = \prod_{\beta=1}^{m} A_\beta$$

Let us now define a new distribution density by the following equation:

$$\bar\rho = \frac{\rho}{\mu_0} \quad .$$

Thus we get

$$\frac{\partial \bar\rho}{\partial t} = \frac{D}{\mu_0 \sqrt{g}} \frac{\partial}{\partial q^i} \left(\mu_0 \sqrt{g}\, Q^{ij} \frac{\partial \bar\rho}{\partial q^j} \right) \tag{5.8}$$

Let us introduce the new set of coordinates by the following equations

$$R^i = R^i(q) \quad , \quad i = 1,\ldots,n-m$$
$$R^{n-m+\alpha} = \phi^\alpha(q) \quad , \quad \alpha = 1,\ldots,m$$

with

$$\frac{\partial R^i}{\partial q^j}\frac{\partial \phi^\alpha}{\partial q^j} = 0 \quad , \quad \begin{array}{l} i = 1,\ldots,n-m \\ \alpha = 1,\ldots,m \end{array}$$

Making this choice the coordinates R^i are intrinsic to the leaves of the foliation given by eq. (5.7).

We also observe that:

$$(g_{ij}) = \left(\begin{array}{c|c} \bar{g}_{ij} & 0 \\ \hline 0 & g_{\alpha\alpha'} \end{array}\right)$$

in the new coordinate system with

$$g_{\alpha\alpha'} = \delta_{\alpha\alpha'} A_\alpha^{-2}$$

Under these assumptions eq. (5.7) takes the form

$$\frac{\partial\bar{\rho}}{\partial t} = \frac{D}{\sqrt{\bar{g}}}\frac{\partial}{\partial R^i}\left(\sqrt{\bar{g}}\,\bar{Q}^{ij}\frac{\partial\bar{\rho}}{\partial R^j}\right) \quad , \quad i,j = 1,\ldots,n-m \tag{5.9}$$

which is the correct diffusion equation for holonomic systems[1]. We therefore conclude that eq. (5.5) has the correct limit for holonomic systems.

We now define as Liouvillian systems those for which there exists a function ψ that satisfies the following equation:

$$Q^{ij}\sum_{\beta=1}^{m} a^k_\beta a^\beta_{j;k} = Q^{ij}\frac{\partial\psi}{\partial q^j} \quad . \tag{5.10}$$

These systems include among them all holonomic system and some non-holonomic systems as well. Its main property is the existence of a measure of probability in phase space given by ψ, independent of the velocities of the system.

In fact, making use of eq. (5.10) nto eq. (5.5) we have

$$\frac{\partial\bar{\rho}}{\partial t} = \frac{D}{\mu}\frac{\partial}{\partial q^i}\left(\mu\, Q^{ij}\frac{\partial\bar{\rho}}{\partial q^j}\right) \tag{5.11}$$

as the diffusion equation for Liouvillian systems. In eq. (5.11) we defined

$$\bar{\rho} = \rho\, e^{-\psi}$$

and

$$\mu = e^{\psi}\sqrt{g}$$

gives the measure of probability in the configuration space of the system.

It is interesting to observe that Liouvillian systems generalise the class of non-holonomic systems discussed in reference 2.

Let us now consider the general case. We set

$$b_j = \sum_{\alpha=1}^{m} a^k_\alpha a^\alpha_{j;k} \tag{5.12}$$

and we observe that in general eq. (5.10) cannot be fulfilled.

We therefore assume now that

$$\rho_0\, Q^{ij} b_j = \rho_0\, Q^{ij}\frac{\partial\psi}{\partial q^j} + C^i \tag{5.13}$$

where ρ_0 is the equilibrium density and C^i is a sourceless vector, i.e. :

$$C^i_{\ ;i} = 0\ . \tag{5.14}$$

Under these assumptions we have

$$\frac{\partial}{\partial q^i}\left[\sqrt{g}\, Q^{ij}\left(\frac{\partial\rho_0}{\partial q^j} - \rho_0\, b_j\right)\right] = 0\ . \tag{5.15}$$

From eqs. (5.13) and (5.15) we get

$$\frac{\partial}{\partial q^i}\left(\sqrt{g}\, Q^{ij}\frac{\partial\rho_0}{\partial q^j} - \rho_0\frac{\partial\psi}{\partial q^j}\right) = 0$$

from where we take

$$\rho_0 = \exp(\psi) \; . \tag{5.16}$$

Taking the divergence on both side of eq. (5.13) and making use of eq. (5.16) we arrive at the equation for ψ:

$$\frac{1}{\sqrt{g}} \frac{\partial}{\partial q^i} \left[\sqrt{g}\, Q^{ij} \left(\frac{\partial \psi}{\partial q^j} - b_j \right) \right] = Q^{ij} \frac{\partial \psi}{\partial q^i} \left(b_j - \frac{\partial \psi}{\partial q^j} \right) \; . \tag{5.17}$$

Therefore, the existence of a positive measure in the general case is connected with the existence of a real solution of eq. (5.17).

Let us write

$$\mu = \rho_0 \sqrt{g}$$

and define $\bar{\rho} = \frac{\rho}{\rho_0}$. Then, eq. (5.5) takes the final form

$$\frac{\partial \bar{\rho}}{\partial t} = \frac{1}{\mu} \frac{\partial}{\partial q^i} \left(\mu Q^{ij} \frac{\partial \bar{\rho}}{\partial q^j} \right) - C^i \frac{\partial \bar{\rho}}{\partial q^i} \tag{5.18}$$

with C^i given by eq. (5.13).

The physical consequences of the general case are very interesting. Similarly to the Liouvillian systems, the non-Liouvillian systems have a non uniform density distribution in its equilibrium state. But, besides that, they also possess at the equilibrium state a pattern of permanent solenoidal currents given by

$$j^i = DC^i \; .$$

Therefore we may say that in general the equilibrium state of non-holonomic is richer in patterns than it is usually observed for statistical systems in equilibrium.

We may further conclude, on physical grounds that, because of theequilibrium, thedirectionof the permanent currents (DC^i) has to be perpendicular to the direction of the gradient of the density of particles. In other words, we must have

$$C^i \frac{\partial\psi}{\partial q^i} = 0 \ .$$

In this case, eq. (5.17) simplifies and we have

$$\frac{1}{\sqrt{g}} \frac{\partial}{\partial q^i} \left[\sqrt{g}\ Q^{ij} \frac{\partial\psi}{\partial q^j}\right] = \frac{1}{\sqrt{g}} \frac{\partial}{\partial q^i} (\sqrt{g}\ Q^{ij} b_j)$$

as the equation for ψ.

6. CONCLUSIONS

Our major conclusion from the study of the diffusion of particles subject to non-holonomical constraints is the appearance of permanent currents in the equilibrium state of the system whose pattern is established exclusively by the constraints.

In order to ensure this result we first derived a Fokker-Planck equation for the diffusion of the system in phase space and established the covariant nature of the equation obtained under general point transformations. Next we developed the whole hierarchy of hydrodynamical equation and showed that in the limit of short mean free paths (when compared with the spatial derivatives of the constraints) we obtain a second order differential equation for the diffusion of particles in configuration space. This equation suggests the classificationof constrained systems as Liouvillians and non-Liouvillians. A Liouvillian system being those that does not exhibit permanent currents in their equilibrium states.

Among the Liouvillian systems are the holonomic systems asthose systems for which the diffusion in configuration space is restricted to the leaves of the foliation resulting from the integrability of the constraint equations.

The second point todrawattention to is the fact that a single equation contains the description of both Liouvillian and non-Liouvillian systems and goes continuously from one class to the other. It also contains the correct limit to holonomic systems.

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