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Relation Between the Classical Wilson Loop and the Angular Parallel Displacement

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We have found a simple relation between the classical Wilson Loop and the angular parallel displacement along a closed path (circle) for the case of the instanton-like potentials.

Encontramos uma relação simples entre o "loop" de Wilson clássico e o deslocamento paralelo angular ao longo de curvas fechadas (círculo) para potenciais tipo instanton.

Lately, the Wilson loop[1,2] has arisen considerable interest in Gauge Theories because it is a gauge invariant quantity and it is thought that it can act as a dynamical variable containing all information about the Gauge Fields[3,7].

In a recent paper, Bollini, Giambiagi and Tiomno[8] computed the classical Wilson loop for the instanton potential by using the spinor representation of the rotation group. In the first part of this paperwe repeat that calculation for the instanton-like potential using the vector representation of the rotation group. This calculation is justified by the fact that in this case we have real rotations and this allows us to bear out the effect of the parallel transport operator, in the in-

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ternal space, for instanton-like potentials, on vectors that have vanishing temporal components.

Now, let us proceed to the computation of the classical Wilson loop in the vector representation of the rotation group.

Consider the instanton-like potentials

$$A_\mu = ia\eta_{\mu\nu} \frac{X_\nu - X_\nu^0}{(X-X^0)^2 + \lambda^2} \tag{1}$$

For each a (real number) we have an instanton-like potential. For $a=1$, e.g., we have the instanton in the vector representation.

The matrices $\eta_{\mu\nu}$ satisfy the following relations,

$$\eta_{ij} = \varepsilon_{ijk}\eta_k \qquad \text{and } \eta_{i4} = \eta_i \tag{2}$$

with

$$\eta_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad ; \eta_2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix} \quad ; \quad \eta_3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} . \tag{3}$$

In the specific case of instanton -like potentials with the center at $X_1^0 = X_2^0 = 0$, we have explicitly for i' on the (X_1,X_2)-plane,

$$A_\mu = ia \frac{\eta_{\mu 1}X_1 + \eta_{\mu 2}X_2 - \eta_{\mu 3}X_3^0 - \eta_{\mu 4}X_4^0}{X + \lambda^2} \tag{4}$$

$$X^2 = X_1^2 + X_2^2 + (X_3^0)^2 + (X_4^0)^2 \tag{5}$$

If we consider now polar coordinates in the (X_1,X_2) - plane and use (2), we find

$$A_\mu dx^\mu = iB(\theta,r)d\theta \tag{6}$$

where

$$B(\theta,r) = -iA(\theta,r) = -\frac{ar^2}{X^2+\lambda^2}\left[\eta_1\left(\cos\theta\,\frac{X_3^0}{r} - \text{sen}\,\frac{X_4^0}{r}\right) + \right.$$

$$\left. + \eta_2\left(\text{sen}\theta\,\frac{X_3^0}{r} + \cos\theta\,\frac{X_4^0}{r}\right) + \eta_3\right]. \qquad (7)$$

In analogy with Bollini, Giambiagi and Tiomno[8], we consider an angle interval $\theta_i \leq \theta \leq \theta_f$, split it into N equal parts $(\theta_f-\theta_i)/N$ and use the relation

$$\exp[iB(\theta)d\theta] = \exp(-i\theta\eta_3)\ \exp[iB(0)d\theta]\ \exp(i\theta\eta_3) \qquad (8)$$

to express the value of $\exp[iB(0)d\theta]$ at the corresponding points $\theta_n = \frac{n}{N}(\theta_f-\theta_i)$ $(n = 1,2,\ldots,N)$. Then, taking the limit $N\to\infty$, the Wilson operator

$$W_{\theta_f\theta_i}(r) = P\exp\left[\int_{\theta_i}^{\theta_f} B(\theta)\,d\theta\right] \qquad (9)$$

is given by

$$W_{\theta_f\theta_i}(r) = \exp(-i\theta_f\eta_3)\exp[i\,(B(0)+\eta_3)(\theta_f-\theta_i)]\exp(i\,\theta_i\eta_3). \qquad (10)$$

By making use of the relation

$$\eta_i^3 = \eta_i \qquad (i = 1,\ 2,\ 3) \quad (\text{Cf. eq.}(3)) \qquad (11)$$

we find

$$W_{\theta_f\theta_i}(r) = \exp(-i\theta_f\eta_3)\left[1+M\ \text{sen}(\theta_f-\theta_i)L + M^2(\cos(\theta_f-\theta_i)L-1)\right]\exp(i\theta_i\eta_3), \qquad (12)$$

where

$$L^2 = B_1^2 + B_2^2 + (B_3+1)^2\ ,\ B(0) = \vec{B}(0).\vec{\eta} \qquad (13)$$

and

$$M = \vec{M}.\vec{\eta}\ ,\ M_1 = \frac{B_1}{L}\ ,\ M_2 = \frac{B_2}{L}\ ,\ M_3 = \frac{B_3+1}{L} \qquad (14)$$

Eq. (12) represents the value of the "integral" (9) for any arc of radius **r** with $8. \leqslant \theta \leqslant \theta_f$.

In particular, for a complete circle ($\theta_i = 0$ and $8_f = 2\Pi$), Eq. (12) reduces to

$$W_{2\pi,0}(r) = 1 + iM \text{ sen}(2\Pi L) + M^2\left[\cos(2\Pi L) - 1\right]. \qquad (15)$$

If we take the trace of Eq. (15) we get the corresponding Wilson Loop, namely,

$$W(r) = \mathrm{Tr}W_{2\pi,0}(r) = 1 + 2\cos(2\Pi L). \qquad (16)$$

Let us now parallel displace three vectors and calculate the cosines of the angles between them and the displaced ones along circles.

Consider the following normalized vectors,

$$\phi_{(1)} = \begin{pmatrix}1\\0\\0\end{pmatrix} \quad ; \quad \phi_{(2)} = \begin{pmatrix}0\\1\\0\end{pmatrix} \quad ; \quad \phi_{(3)} = \begin{pmatrix}0\\0\\1\end{pmatrix} \qquad (17)$$

If we pa'rallel displace each of the vectors along a circle, the final vectors are given by

$$\phi'_{(i)} = W_{2\pi,0} \; \phi_{(i)} \qquad (18)$$

where

$$W_{2\pi,0} = P \exp\left\{\int_0^{2\pi} B(\theta)d\theta\right\} \qquad (19)$$

is the parallel displacement operator[9] (along circles).

As the vectors $\phi_{(i)}$ and $\phi'_{(i)}$ have vanishing temporal components, wecandefinereal angles[9] $a_{(i)}$ bymeansof

$$\cos \alpha_{(i)} = \tilde{\phi}_{(i)} \, W_{2\pi,0} \; \phi_{(i)} \qquad (20)$$

In Eq. (20) we have used the facts that $\phi_{(i)}$ and $\phi'_{(i)}$ are normalized to one and that the metric has signature (+,+,+).

In our notation $\tilde{\phi}_{(i)}$, is the transpose of $\phi_{(i)}$.

By using Eqs. (15) and (20) we get

$$\cos \alpha_{(i)} = \tilde{\phi}_{(i)} \left[1 + iM \operatorname{sen}(2\Pi L) + M^2(\cos(2\Pi L) - 1)\right]\phi_{(i)} \quad (21)$$

A calculation of $\cos \alpha_{(i)}$ $(i = 1,2,3)$ gives

$$\cos \alpha_{(1)} = 1 + \frac{1}{L^2}\left[B_2^2 + (B_3+1)^2\right](\cos(2\Pi L) - 1) \quad (22)$$

$$\cos \alpha_{(2)} = 1 + \frac{1}{L^2}\left[B_1^2 + (B_3+1)^2\right](\cos(2\Pi L) - 1) \quad (23)$$

$$\cos \alpha_{(3)} = 1 + \frac{1}{L^2}(B_1^2 + B_2^2)(\cos(2\Pi L) - 1) \quad (24)$$

Adding (22), (23) and (24) we find

$$\sum_{i=1}^{3} \cos \alpha_{(i)} = 1 + \cos(2\Pi L) \quad (25)$$

Now, comparing (25) and (16), we conclude that

$$W = \operatorname{Tr} W_{2\pi,0} = \sum_{i=1}^{3} \cos \alpha_{(i)} \quad (26)$$

Eq. (26) is the announced relation between the classical Wilson Loop and the angles of parallel displacement.

Notice that Eq. (26) is valid independently of the value of a, i.e., for the potentials of the instanton, meron and general instanton-like potentials. Observe, however, that the values of the angles are dependent on a. We know that the values of these angles are a measure of the mean curvature. Thus the field of the instanton, for example, "bends" space in a different way from that of the field of merons and other instanton-like configurations.

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REFERENCES

1. K.Wilson, Phys. Rev. D*10*, 2445 (1974).
2. S.Mandelstam, Phys.Rev. 175, 1580 (1968).
3. J.L.Gervais, A.Neveu, Nucl.Phys. B*153*, 445 (1979).
4. J.L.Gervais, M.T. Jaekel, A.Neveu, Nucl.Phys. B *155*, 75 (1979).
5. A.M.Polyakov, Phys.Lett. *82*B, 247 (1979).
6. Y.M.Makeenko, A.A.Migdal, Preprint I.T.E.P., 86 (1979).
7. M.A.Virasoro, Phys.Lett. 82 B, 436 (1979).
8. C.B.Bollini, J.J.Giambiagi, J.Tiomno, Nuovo Cim. *59*A, 412 (1980).
9. C.G.Bollini, J.J.Giambiagi, J.Tiomno, Preprint C.B.P.F., NF 010/81, (1981).