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Self-Graviting Null Fluids

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An algorithm for generating solutions to the Einstein equations with a null fluid source, from vacuum solutions, is studied for space-times that admit an Abelian group of motions G_2 that acts orthogonally transitively.

Estuda-se um algoritmo para gerar soluções das equações de Einstein com fonte de fluido nulo a partir de soluções para o vácuo, nos espaço; tempos que admitem um grupo Abeliano de movimentos G_2 que atua ortogonalmente e transitivamente.

In a recent paper[1] an algorithm to generate solutions to Einstein equations coupled to a massless scalar field (or a perfect fluid with $p = w$ equation of state) from solutions to vacuum Einstein equations was studied for the metric

$$ds^2 = e^{2k}(dt^2 - dx^2) - r[f(dy + q\ dz)^2 + (1/f)dz^2] , \qquad (1)$$

where k, r, f and q are functions of t and a only. This metricadmits an Abelian group of isometries G_2 that acts orthogonally transitively[1].

In this note we show that a similar algorithm exist, thatgenerates solutions to Einstein field equations coupled to a null fluid fromsolutionsof vacuum Einstein equations for the samemetric (1). Particular cases of (1) have been widely used to study gravitational waves withoneor twodegreesof freedom[2,3]. Solutions to Einstein equations coupled to a null fluid source are of some physical interest because they can be interpreted as metrics generated by shining objects[4], i.e., objects emitting directed flows of electromagnetic radiation[5].

Let us prove the following solution generating theorem:

Einstein field equations with a null fluid source for the metric obtained by replacing k by $k + \Omega$ in (1), i.e.,

$$R_{ab}(k + \Omega,\ r,\ f,\ q) = -w\ \ell_a\ \ell_b, \qquad \ell_a \ell^a = 0 \tag{2}$$

are equivalent to Einstein vacuum field equations for the metric (1), i.e.,

$$R_{ab}(k,\ r,\ f,\ q) = 0 \tag{3}$$

where Ω is a function determined by a single quadrature.

To prove the theorem we first notice that for metrics like (1) we have $R_{ty} \equiv R_{tz} \equiv 0$. Thus Eq. (2) implies that $\ell_Y \equiv \ell_z \equiv 0$. Hence, there is no loss of generality in choosing $\ell^a = (1,\ \pm 1,\ 0,\ 0\ )$. The explicit form of the Ricci tensor for metrics like (1) can be found in references 1, 2 and 6. The contracted Bianchi identities for (2) give us

$$w_0 \pm w_1 + w|4(k_0 \pm k_1) + (r_0 \pm r_1)/r| = 0\ , \tag{4}$$

where the subscripts 0 and 1 indicate differentiation with respect to t and x respectively. The general solution to (5) is

$$w = (A/r)\ \exp\ (-4k)\ , \tag{5}$$

where $A = A(t \mp x)$

Now the Einstein equations (2) are equivalent to:

$$2\left[(k + \Omega)_0\ r_0 + (k + \Omega)_1\ r_1\right]/r - 2r_{00}/r + \frac{1}{2}\ (r_0^2 + r_1^2)/r^2$$

$$- \frac{1}{2}\ (f_0^2 + f_1^2)/f^2 - \frac{1}{2}\ f^2 (q_0^2 + q_1^2) = -\ 2A/r\ - \tag{6a}$$

$$- 2\left[(k+\Omega)_{00} - (k+\Omega)_{11}\right] + \frac{1}{2}(r_0^2 - r_1^2)/r^2 -$$

$$- \frac{1}{2}(f_0^2 - f_1^2)/f^2 - \frac{1}{2} f^2 (q_0^2 - q_1^2) = 0 \,, \tag{6b}$$

$$\left[(k+\Omega)_1 \, r_0 + (k+\Omega)_0 \, r_1\right]/r + \frac{1}{2} r_0 \, r_1/r^2 - r_{01}/r -$$

$$- \frac{1}{2} \, f_0 f_1/f^2 - \frac{1}{2} f^2 \, q_0 \, q_1 = \pm A/r \tag{6c}$$

$$(f_{00} - f_{11})/f + (f_0 \, r_0 - f_1 \, r_1)/(fr) - (f_0^2 - f_1^2)/f^2 -$$

$$- f^2 (q_0^2 - q_1^2) = 0 \,, \tag{6d}$$

$$q_{00} - q_{11} + (q_0 \, r_0 - q_1 \, r_1)/r + 2(f_0 \, q_0 - f_1 \, q_1)/f = 0 \tag{6e}$$

$$r_{00} - r_{11} = 0 \,. \tag{6f}$$

The conventions used in this note are those of reference 7 except that we have taken $c = 8\pi G = 1$.

Choosing a function R such that,

$$r_0 \, \Omega_0 + r_1 \, \Omega_1 = - A \tag{7a}$$

$$r_1 \, \Omega_0 + r_0 \, \Omega_1 = \pm A, \tag{7b}$$

$$\Omega_{00} - \Omega_{11} = 0 \,, \tag{8}$$

the system of equations (6) is equivalent to the Einstein equations (3). Equations (8) follow from (6f) and (7), and equations (7) give us

$$\Omega = - \int A(t \mp x) \, \frac{d(t \mp x)}{r_0 \mp r_1} \tag{9a}$$

$$r_0^2 \neq r_1^2 \tag{9b}$$

Q.E.D.

We notice that if $(g_{\mu\nu},A)$ is a solution to (2) then $(Ag_{\mu\nu},A)$ is also a solution, whenever X is a constant, since we can always add a constant to R and rescale the coordinates y and z. We also havethat the rôles of t and x can be interchanged by adding $i\pi$ to R. In the presente note, we shall consider line elements related by the above mentioned transformations as equivalent.

Letting $q \to 0$ and $f \to 1$ in (6), solving the differential equations and making a change of variables, we find

$$ds^2 = \frac{e^{2\Omega}}{\pm\sqrt{t}} (dt^2 - dx^2) - t(dy^2 + dz^2) \ . \tag{10a}$$

$$\Omega = - \int A(t \mp x)\ d(t \mp x) \tag{10b}$$

The metric (1) with $q = 0$ and $f = 1$ is the rnost genera' metric that admits the three parameter group of symrnetries that characterizes plane symmetry[S]. Thus, the solution (10) is the general solution for a plane symmetric self-graviting null fluid. This solution is a special caseof the one discussed in reference 8.

Letting $k \to \omega - X + R$, $f \to e^{2\lambda}/x$, $q \to \chi$ and $r \to x$ in (1) and (9) we get

$$ds^2 = e^{2(\omega - \lambda + \Omega)}(dt^2 - dx^2) - e^{2\lambda}(dy + \chi' dz)^2 -$$

$$- x^2\ e^{-2\lambda}\ dz^2 \tag{11a}$$

$$\Omega = \int A(t \mp x)\ d(t \mp x) \ .$$

The field equations (6) for the present case have been studied by many authors. See for instance reference 10 and references therein. The space-time (11) represents a self-graviting null fluid with cylindrical symmetry and no reflection symmetry. Making $\chi \to 0$ in (IIa) we recover the wellknown Einstein - Rosen metric[11].

Letting $k \to \omega - X + R$, $f \to x\ e^{-2X}$, $q \to 0$ and $r \to xt$ in (1) and (9) we find

$$ds^2 = e^{2(\omega - \lambda + \Omega)}(dt^2 - dx^2) - t(x^2 e^{-2\lambda} dy^2 + e^{2\lambda} dz^2) \tag{12a}$$

$$\Omega = - \int A(t \mp x) \frac{d(t \mp x)}{x \mp t} \tag{12b}$$

The metric (11) represents the metric of a self-graviting null fluid with cylindrical symrnetry and reflection symmetry. Note that this metric is a yeneralization of Einstein - Rosen metric. The equations (6) for this case have been studied in references 12 and 13.

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