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The Propagator for an Oscillator with Time-Dependent Frequency

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The propagator for a one-dimensional oscillator with time dependent frequency is calculated by the path-integral method.

O propagador para um oscilador unidimensional de frequência dependente do tempo é calculado pelo método das integrais de caminho.

1. INTRODUCTION

The propagator between the states $|x',t'>$ and $|x'',t''>$ of a one-dimensional quantum system characterized by a Hamiltonian

$$H = \frac{p^2}{2m} + V(x,t) \qquad (1)$$

is given by[1].

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$$K(x'',t'';x',t') =$$

$$\lim_{n\to\infty} \int \prod_{i=1}^{n} dx_i \prod_{j=1}^{n+1} \left(\frac{dp_j}{2\hbar}\right) \exp\left\{\frac{i}{\hbar}\left[\sum_{k=1}^{n+1} p_k(x_k - x_{k-1}) - \varepsilon H_k\right]\right\} \qquad (2)$$

where

$$x_i = x(t_i), \quad t_0 = t', \quad x_0 = x', \quad x_{n+1} = x'', \quad \varepsilon = \frac{t''-t'}{n+1}, \quad t_{i+1} = t_i\ \varepsilon$$

(3)

and $$H_k = \frac{p_k^2}{2m} - V[x(t_{k,k-1}), t_{k.k-1}] \qquad (4)$$

where $t_{k,\,k-1}$ is any value of t belonging to the interval $[t_{k-1}, t_k]$ (*)

The explicit calculation using the form (2) is limited to some special cases due to technical problems in the evaluation of the integrals involved.

One of these is the one-dirnensional harrnonic oscillator with time-dependence frequency for which

$$V(x,t) = \frac{m}{2}\,\omega^2(t)\, x^2 \ . \qquad (5)$$

As is well known, in this case the propagator is given by A $e^{S_{cl}}$ where A is the normalization factor and S_{cl} is the classical action. So it is interesting to reproduce, by the path-integral method, this result. The step-by-step integration inherent to this method leads to an expression for the propagator involving four coefficients. A calculation of these coefficients has already been performed by Khandekar and Lawande[2]. Three of them are easily obtained by transforming the finite -difference equations they obey into differential equations, however

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(*) Usually $x(t_{k,k-1})$ is taken as the mean-value between x_k and x_{k-1}. But to order ε, one may take for $t_{k,k-1}$ any value of the interval $[t_{k-1}, t_k]$ without changing the integral, what amounts to take any value for x in the interval $[x_{k-1}, x_k]$.

the fourth coefficient satisfies a much more complicated equations and its solution was not derived by the cited authors. The major goal we pursued in the present article is to obtain rigorously the fourth coefficient, called p_n by Khandekar and Lawande and C_n here.

The strategy used was to rewrite the equation for C_n as a truncated continuous fraction just like the equation for A_n, oneof the other coefficients. Then we realized that C_n and A_n were related by time-reversal and thus the solution for C_n followed from that already obtained for A_n.

Hence we showed rigorously that the path-integral method gives the correct expression for the propagator of the oscillator with a time -dependend frequency.

The details of calculation are presented in next section.

2. CALCULATION OF THE PROPAGATOR

We now consider the evaluation of the integral in eq. (2) for the special case of the potential of eq. (5). Choosing $t_{k\, k-1} = \frac{t_k + t_{k-1}}{2}$, eq. (4) becomes

$$H_k = \frac{p_k^2}{2m} - \frac{m}{2}\,\omega_{k,k-1}^2 \left(\frac{x_k + x_{k-1}}{2}\right)^2 \tag{6}$$

where

$$\omega_{k\, k-1} = \omega(t_{k,k-1}) \tag{7}$$

By the repeated use of the Fresnel integral

$$\int_{-\infty}^{\infty} \frac{dp}{2\pi} \exp\left|i\varepsilon\left(p\,\dot{x} - \frac{p^2}{2}\right)\right| = (2\pi\, i\varepsilon)^{-1/2} \exp\left(\frac{i\varepsilon\,\dot{x}^2}{2}\right) \tag{8}$$

we get after n integrations

$K(x'',t'';x',t') =$

$$= \left(\frac{m}{2i\hbar}\right)^{1/2} \lim_{n\to\infty} \left\{\left(\frac{1}{\varepsilon N_n}\right)^{1/2} \exp\left[\frac{im}{2\varepsilon\hbar}\left(A_n x''^2 - 2B_n x'' x' + C_n x'^2\right)\right]\right\} \tag{9}$$

It what follows we will show the expressions for N_n, A_n, B_n and C_n and calculate their limit when $n \to \infty$.

Let us start defining

$$P^{\pm}_{k,k-1} = 1 \pm \frac{\varepsilon^2}{4} \omega^2_{k,k-1} \tag{10}$$

The normalization factor N_n then satisfies the recurrence relation

$$N_k = \left(P^-_{k+1,k} + P^-_{k,k-1}\right) N_{k-1} - P^{+2}_{k,k-1} N_{k-2} \tag{11}$$

with the initial conditions

$$N_{-1} = 0 \quad \text{and} \quad N_0 = 1 \tag{12}$$

Let $M(t_{k+1}) = \varepsilon N_k$. (13)

Then, taking the limit $n\to\infty$, the finite-difference equation (11) and the initial conditions (12) take the following forms respectively:

$$\ddot{M} + \omega^2(t) M = 0 \tag{14}$$

$$M(t') = 0 \quad \text{and} \quad \dot{M}(t') = 1 \tag{15}$$

A solution of the form

$$M(t) = s(t)\sin(\gamma(t) - \gamma(t')) \tag{16}$$

automatically fulfils the conditions $M(t')=0$. Also eq. (14) is satisfied if

$$\ddot{s} - s\dot{\gamma}^2 + \omega^2 s = 0 \tag{17}$$

and

$$s^2\dot{\gamma} = \text{constant} \tag{18}$$

The remaining boundary condition is satisfied if

$$s(t')\dot{\gamma}(t') = 1 \tag{19}$$

The equations (17)-(19) can be recast in the form

$$\ddot{s} - \frac{s^2(t')}{s^3} + \omega^2 s = 0 \tag{20}$$

and

$$s^2\dot{\gamma} = s(t') \tag{21}$$

From (13) we have

$$\lim_{n\to\infty} \varepsilon N_n = M(t'') = s(t'')\sin(\gamma(t'') - \gamma(t')) \ , \tag{22}$$

and using (19) and (21) we get

$$\lim_{n\to\infty} \varepsilon N_n = M(t'') = \frac{\sin[\gamma(t'') - \gamma(t')]}{(\dot{\gamma}(t'')\dot{\gamma}(t'))^{1/2}} \ . \tag{23}$$

Let us now turn to coefficient A_n. After n integrations it appears as:

$$A_n = P^-_{n+1,n} - \cfrac{P^{+2}_{n+1,n}}{P^-_{n+1,n} + P_{n,n-1} - \cfrac{P^{+2}_{n,n-1}}{(P^-_{n,n-1} + P^-_{n-1,n-2}) - \cdots - \cfrac{P^{+2}_{2,1}}{P^-_{2,1} + P^-_{1,0}}}} \tag{24}$$

Using relation (10) we can express A_n in terms of the normalization factor N_n:

$$A_n = \frac{P^2_{n+1,n} N_n - P^2_{n+1,n} N_{n-1}}{N_n} \qquad (25)$$

in the limit $n \to \infty$

$$\frac{A_n}{\varepsilon} = \frac{\dot{M}(t'')}{M(t'')}$$

$$= \frac{\dot{s}(t'')}{s(t'')} + \dot{\gamma}(t'')\operatorname{cotg}[\gamma(t'') - \gamma(t')] \qquad (26)$$

Next, the coefficient B_n can be written as

$$B_n = \frac{P^+_{n+1,n} P^+_{n,n-1} \cdots P^+_{1,0}}{N_n} \qquad (27)$$

so that from (23)

$$\lim_{n\to\infty} \frac{B_n}{\varepsilon} = \frac{1}{M(t'')} \qquad (28)$$

Finally the coefficient C_n is given by

$$C_n = P^-_{1,0} - \frac{P^{+2}_{1,0}}{N_1} - \frac{P^{+2}_{1,0} P^{+2}_{2,1}}{N_1 N_2} - \frac{P^{+2}_{3,0} P^{+2}_{2,1} P^{+2}_{3,2}}{N_2 N_3}$$

$$- \cdots - \frac{P^{+2}_{1,0} P^{+2}_{2,1} \cdots P^{+2}_{n,n-1}}{N_{n-1} N_n}$$

$$P^-_{1,0} - \frac{P^{+2}_{1,0}}{N_1}\left(1 + \frac{P^{+2}_{2,1} N_0}{N_2}\left(1 + \frac{P^{+2}_{3,2} N_1}{N_3}\left(1 + \cdots + \frac{P^{+2}_{n,n-1} N_{n-2}}{N_n}\right)\cdots\right)\right)$$

$$(29)$$

To solve directly this equation in a rigorous way is a difficult task. As we said in Introduction, we will put C_n in a form similar to A_n in eq. (24), i.e., transform the series (29) into a truncated continuous fraction. To this aim, the trick consists in defining the transformation

$$s_p(\omega) = 1 + a_p\omega \quad \text{with} \quad p = 1,2 \ldots \tag{30}$$

and applyiny successively this kind of transformation and taking $\omega=0$, so that:

$$s_1 s_2 \ldots s_n(0) = 1 + a_1(1 + a_2(1 + \ldots + a_{n-1}) \ldots) \tag{31}$$

But remark that eq. (30) is equivalent to

$$s_p(\omega) = \cfrac{1}{1 + \cfrac{a_p}{a_p + \cfrac{1}{\omega}}}, \qquad p = 1, 2, \ldots \tag{32}$$

Then

$$s_1 s_2 \ldots s_n(0) = \cfrac{1}{1 - \cfrac{a_1}{a_1 + 1 - \cfrac{a_2}{a_2 + 1 - \ddots - \cfrac{a_{n-1}}{a_{n-1} + 1}}}} \tag{33}$$

Eqs. (31) and (33) allows us to convert the series into a continuous fraction. Now, using this fact and the recurrence relation (10), eq. (29) reads

$$C_n = P^-_{1,0} - \cfrac{P^{+2}_{1,0}}{P^-_{2,1} + P^-_{1,0} - \cfrac{P^{+2}_{2,1}}{P^-_{3,2} + P^-_{2,1} - \ddots - \cfrac{P^{+2}_{n,n-1}}{P^-_{n+1,n} - P^-_{n,n-1}}}} \tag{34}$$

This complicated looking expression can be evaluated very simply.

By comparing (34) and (24) we see that C_n is exactly what we obtain by reversing time in expression (24), i.e., by making the following exchanges

$$0 \leftrightarrow n+1 \quad , \quad 1 \leftrightarrow n, \ \ldots$$

Thus in the limit $n\to\infty$ we obtain C_n by taking A_n as in (26) and exchanging t'' and t' and keeping in mind that the exchange above implies $\frac{d}{dt} \to -\frac{d}{dt}$.

Hence when $n\to\infty$,

$$\frac{C_n}{\varepsilon} = -\frac{\dot{s}(t')}{s(t')} + \dot{\gamma}(t') \,\mathrm{cotg}\left[\gamma(t'') - \gamma(t')\right] \tag{35}$$

This gives the final result for the propagator as

$$K(x'',t'';x',t') = \left\{\frac{m}{2\pi i\hbar} \cdot \frac{\left[(\dot{\gamma}(t'')\dot{\gamma}(t')\right]^{1/2}}{\sin\left[\gamma(t'') - \gamma(t')\right]}\right\}^{1/2}$$

$$\exp \frac{im}{2\hbar}\left\{\left[\frac{\dot{s}(t'')}{s(t'')} + \dot{\gamma}(t'') \,\mathrm{cotg}\left[\gamma(t'') - \gamma(t')\right]\right] x''^2\right.$$

$$+ \left[-\frac{\dot{s}(t')}{s(t')} + \dot{\gamma}(t') \,\mathrm{cotg}\left[\gamma(t'') - \gamma(t')\right]\right] x'^2$$

$$\left. - 2\,\frac{\left[\dot{\gamma}(t'')\dot{\gamma}(t')\right]^{1/2}}{\sin\left[\gamma(t'') - \gamma(t')\right]}\, x''\, x' \right\} \tag{36}$$

Where $s(t)$, $\gamma(t)$ are to be obtained by solving eqs. (20) and (21) once $\omega(t)$ given.

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