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Quasi-Potential Equations in a Simple Context

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The so called "Quasi-Potential equations" are explained in the simple context of the non-relativistic Bethe-Salpeter Equation. The many ways of dealing with retardation effects appear very clearly in the present context as the responsible for the possibility of having an infinite number of "on shell equivalent" such equations.

As assim chamadas equações quasi-potenciais são explicadas usando a equação não relativística de Bethe-Salpeter. O fato de existir um número infinito de equações deste tipo, todas equivalentes "on shell" aparece muito claramente no presente contexto.

1. INTRODUCTION

In a previous paper[1] one of us argued, among other things, that a consideration of the non-relativistic Bethe-Salpeter* equation could help students to understand the vast literature on the so called "Quasi-potential equations"[2] (also called three-dimensional relativistic equations[3]). Such three-dimensional equations have been applied suc-

* By non-relativistic we mean Galilean Invariant. By Relativistic we mean Einstenian Invariant.

cessfully in problems ranging from bound state problems in Quantum Electrodynamics[4] to intermediate energy nuclear physics[5].

The variety of quasi-potential equations is often astounding if one does not understand that different equations merely represent different "off-the-mass-shell" extrapolations of the two particle scattering amplitudes. This was shown very clearly by A.Klein and T.H.Lee[6] who showed how to derive from the Relativistic Bethe-Salpeter equation, the Gross equation[7], the Fronsdal equation[8] and the Levy-Macke-Klein[9] equation.

In this paper we apply the A.Klein and T.H. Lee method to the non-relativistic Bethe-Salpeter equation. The pedagogical advantage of doing this is that we have simpler non-relativistic kinematics andavoid positive energy projections and hence y-matrix complications.

In the second section we use the non-relativistic Bethe-Salpeter equation to introduce our notation and to show that if one assumes the interaction between the particles as instantaneous then we have a reduction to the Lippmann-Schwinger equation.

In the third section we explain and apply the method described in ref.5 to the non-relativistic Bethe-Salpeter equation.

Finally in the fourth section we briefly review the problem of obtaining a relativistic three-dimensional three body formalismwhich have not been solved satisfactorily. It is suggested that consideration of the three-body non-relativistic Bethe-Salpeter equation could be helpfull in claryfying this problem.

2. THE NON RELATIVISTIC BETHE-SALPETER EOUATION

In this paper we consider a simple system: a two particle system whose hamiltonian is given by

$$H = \frac{\vec{P}_1^2}{2m_1} \frac{\vec{P}_2^2}{2m_1} + V(\vec{r}_1 - \vec{r}_2) = H_0 + V(\vec{r}_1 - \vec{r}_2) = H_0 + V \qquad (1)$$

where $\vec{P}_1(m_1)$ and $\vec{P}_2(m_2)$ are the momentum (masses) of particles 1 and 2 and $V(\vec{r}_1-\vec{r}_2)$ is the interaction between then. Although only a dependence of $\vec{r}_1-\vec{r}_2$ is indicated in equation (1), V can depend on other variables such as spin.

If we decompose the S-matrix as

$$<f|S|i> = \delta_{fi} + <f|T|i> \tag{2}$$

then $<f|T|i>$ satisfy the following Non-Relativistic Bethe - Salpeter Equation

$$<\vec{P}_1E_1\vec{P}_2E_2|T|\vec{P}_1'E_1'\vec{P}_2'E_2'> = <\vec{P}_1E_1\vec{P}_2E_2|\tilde{V}|\vec{P}_1'E_1'\vec{P}_2'E_2'> +$$

$$+ \iint \frac{d^3P_1''}{(2\pi)^3}\frac{d^3P_2''}{(2\pi)^3}\frac{dE_1''}{(2\pi)}\frac{dE_2''}{(2\pi)} <\vec{P}_1E_1\vec{P}_2E_2|\tilde{V}|P_1''E_1''P_2''E_2''>$$

$$\times \frac{i}{\frac{\vec{P}_1''^2}{2m_1} - E_1'' + \varepsilon i}\;\frac{i}{\frac{\vec{P}_2''^2}{2m_2} - E_2'' + i\varepsilon} <\vec{P}_1''E_1''\vec{P}_2''E_2''|T|\vec{P}_1'E_1'\vec{P}_2'E_2'> \tag{3}$$

where the Bethe-Salpeter kernel $<\vec{P}_1E_1\vec{P}_2E_2|\tilde{V}|\vec{P}_1'E_1'\vec{P}_2'E_2'>$ is given, if the interaction is instantaneous (see ref. 1), by

$$<\vec{P}_1E_1\vec{P}_2E_2|\tilde{V}|\vec{P}_1'E_1'\vec{P}_2'E_2'> = -2\pi i\ \delta(E_1+E_2-E_1'-E_2')<\vec{P}_1\vec{P}_2|\hat{V}|\vec{P}_1'\vec{P}_2'> \tag{4}$$

The product

$$\frac{1}{\frac{\vec{P}_1''^2}{2m_1} - E_1 + i\varepsilon} \times \frac{1}{\frac{\vec{P}_2''^2}{2m_2} - E_2 + i\varepsilon}$$

is refered to as the propagator of the Bethe-Salpeter Equation.

Equation (3) describes the scattering of particles which are "off the nass shell", that is, $\vec{P}_1^2 \neq 2m_1E_1$, etc. The physical amplitude is obtained by putting the particles "on the mass shell", that is, by taking $\vec{P}_1^2 = 2m_1E_1$, etc.

The form of the interaction (4) suggests that T has the form

$$<\vec{P}_1 E_1 \vec{P}_2 E_2 | T | \vec{P}_1' E_1' \vec{P}_2' E_2'> = -2\pi i \;\delta(E_1+E_2-E_1'-E_2') <\vec{P}_1 \vec{P}_2 | \hat{T} | \vec{P}_1' \vec{P}_2'> . \quad (5)$$

Replacing (5) in (2), and integrating in the energy variables we get for $<\vec{P}_1\vec{P}_2|\tilde{T}|\vec{P}_1'\vec{P}_2'>$ the result

$$<\vec{P}_1 \vec{P}_2 | \hat{T} | \vec{P}_1' \vec{P}_2'> = <\vec{P}_1 \vec{P}_2 | \hat{V} | \vec{P}_1' \vec{P}_2'> + \iint \frac{d^3P_1''}{(2\pi)^3} \frac{d^3P_2''}{(2\pi)^3} <\vec{P}_1 \vec{P}_2 | \hat{V} | \vec{P}_1'' \vec{P}_2''>$$

$$\frac{i}{\frac{\vec{P}_1''^2}{2m_1} + \frac{\vec{P}_2''^2}{2m_2} - (E_1+E_2) + i\varepsilon} < \vec{P}_1'' \vec{P}_2'' | \hat{T} | \vec{P}_1' \vec{P}_2' > \quad (6)$$

That is, the Bethe-Salpeter equation reduces to the Lippmann- Schwinger. To indicate that $<\vec{P}_1\vec{P}_2|\hat{T}|\vec{P}_1'\vec{P}_2'>$ depends on the variable E_1+E_2 we rewritte (5) as

$$<\vec{P}_1 E_1 \vec{P}_2 E_2 | T | \vec{P}_1' E_1' \vec{P}_2' E_2'> = -2\pi i \;\delta(E_1+E_2-E_1'-E_2') <\vec{P}_1 \vec{P}_2 | T|(E_1+E_2)| \vec{P}_1' \vec{P}_2'>$$

(7)

3. TRI-DIMENSIONAL EQUATIONS DERIVED FROM THE NON-RELATIVISTIC BETHE-SALPETER EOUATION

In field theory however, the reduction rnentioned above does not take place[10], that is, we do not have a Lipprnann-Schwinger equation and have to be content with the Bethe-Salpeter equation or a "Quasi-Potential" Equation.

The reason why it is not possible to have a Lippmann- Schwinger equation can be seen clearly if we transform to other variables

$$\vec{P} = \vec{P}_1 + \vec{P}_2 \qquad E = E_1 + E_2$$
$$\vec{q} = \frac{m_2\vec{P}_1 - m_1\vec{P}_2}{m_1 + m_2} \qquad \omega = \frac{m_2E_1 - m_1E_2}{m_1 + m_2} \quad (8)$$

and similarly for the primed quantities. The variable ω is called "relative energy". It is the appearance of this variable that does not allow the reduction expressed in equation (6) to take place, and gives rise to the so called "spurious solutions" to the Bethe-Salpeter equation[11].

Invariance with respect to space and time translation allows[12] one to writte

$$<\vec{P}_1E_1\vec{P}_2E_2|\tilde{V}|\vec{P}_1'E_1'\vec{P}_2'E_2'> = -(2\pi)^4 i\ \delta(E-E')\delta(\vec{P}-\vec{P}')\ <\vec{q}\omega|\hat{V}|q'\omega'> \qquad (9)$$

(if the interaction is assumed instantaneous then we would have

$$q\omega\ \hat{V}\ q'\omega' = <q\ V\ q'\ >)\ .$$

The form (8) for the interaction suggests

$$<PqE\omega|T|P'q'E'\omega'> = -(2\pi)^4\ i\ \delta^3(P-P')\delta(E-E')\ <q\omega|\hat{T}|q'\omega'> \qquad (9)$$

and replacing (9) and (10) in (3) and using (8) we get the non relativistic Bethe-Salpeter Equation in "relative variables"

$$<q\omega|\hat{T}|q'\omega'> = <q\omega|\hat{V}|q'\omega'> + \int \frac{d^3q''}{(2\pi)^3}\ \frac{d\omega''}{(2\pi)}$$

$$<q\omega|\hat{V}|q''\omega''> \frac{i}{\left[\frac{1}{2m_1}\left(\frac{m_1}{m_1+m_2}\vec{P}+q''\right)^2 - \frac{m_1}{m_1+m_2}E - \omega'' + i\varepsilon\right]} \times$$

$$\times \frac{i}{\left[\frac{1}{2m_2}\left(\frac{m_2}{m_1+m_2}\vec{P}-q''\right)^2 - \frac{m_2}{m_1+m_2}E + \omega'' + i\varepsilon\right]} \times <q''\omega''|\hat{T}|q'\omega'> \qquad (10)$$

Again if $<\vec{q}\omega|\hat{V}|\vec{q}'\omega'> = <\vec{q}|\hat{V}|\vec{q}'>$, that is, if the interaction is instantaneous then the integration in ω' can be performed and we reçain the Lippmann-Schwinger equation.

In the Relativistic Bethe - Salpeter equation however,

$\langle q\omega|\hat{V}|q'\omega'\rangle$ is given by the sum of the two particle irreducible Feynman diagrams and depends explicitly on ω and ω'. Equation (10) is in this case a four dimensional equation instead of three-dimensional as the Lippmann-Schwinger equation.

In the center of mass system $\vec{P}=0$ so that equation (10) simplifies considerably. From now on we shall therefore assume to be in the center of mass frame. To obtain additional simplification we shall also assume from now on that $m_1=m_2=m$. With this simplification (10) becomes

$$\langle q\omega|\hat{T}|q'\omega'\rangle = \langle q\omega|\hat{V}|q'\omega'\rangle + \int \frac{d^3q''}{(2\pi)^3}\frac{d\omega''}{(2\pi)}\langle q\omega|\hat{V}|q''\omega''\rangle \times$$

$$\times \frac{i}{\left(\frac{q''^2}{2m} - \frac{1}{2}E - \omega'' + i\varepsilon\right)}\frac{i}{\left(\frac{q''^2}{2m} - \frac{1}{2}E + \omega'' + i\varepsilon\right)}\langle q''\omega''|\hat{T}|q'\omega'\rangle \tag{11}$$

The physical amplitude is obtained from (11) by making E equal to the total initial or final kinetic energy.

Although equation (11) is intrinsicaly four dimensional it is possible to derive an approximate three-dimensional integral equation (hence, Lippmann-Schwinger like) which for this reason is called quasi--potential or three-dimensional "relativistic" equation. The procedure however is not unique and in fact an infinite number of equations result, all of which however, give the same "on shell amplitudes" if solved exactly.

To explain how this isdonewe first writteequation (11) in simbolic form as

$$\hat{T} = \hat{V} + \hat{V}\,G\,\hat{T} \tag{12}$$

Equation (12) is equivalente to another equation where the interaction $\hat{V}$ is replaced by v and the propagator G by g

$$\hat{T} = v + v\,g\,\hat{T} \tag{13}$$

if v and g satisfies

$$v = \hat{V} + \hat{V}\,(G - g)\, v \qquad (14)$$

This can be shown solving for v in (14) and replacing in (13).

We now define g as

$$g = \overleftarrow{P}\, \frac{1}{\frac{q''^2}{m} - E + i\varepsilon}\, \vec{P} \qquad (15)$$

where $\vec{P}$ is e projection operator thatfixes the value of the ω variable to a certain value. For example $\vec{P}$ can be taken simply as

$$\vec{P}(\;) = \int d\omega\; \delta(\omega)\,(\;) \qquad (16)$$

whichfixes the value ω=0 which is the value of ω "on the mass shell" in the center of mass frame.

To see how the procedure works we note that if we project equations (13) left and right we get

$$\langle\vec{q}\omega{=}0\,|\hat{T}|\vec{q}\,'\omega = 0\rangle = \langle\vec{q}\omega{=}0\,|v|\vec{q}\,'\omega{=}0\rangle + \int \frac{d^3q''}{(2\pi)^3}\, \langle q\,'\omega{=}0\,|v|q\,''\omega{=}0\rangle$$

$$\frac{1}{\frac{q''^2}{m} - E + i\varepsilon}\; \langle\vec{q}\,''\omega{=}0\,|\hat{T}|\vec{q}\,''\omega{=}0\rangle \qquad (17)$$

which is a Lippmann-Schwinger like equation.

To get $\langle q\omega{=}0|v|q\,'\omega{=}0\rangle$ we have to solve equation (14) because if we project it before solving we get

$$\langle\vec{q}\omega{=}0\,|v|\vec{q}\,'\omega{=}0\rangle = \langle\vec{q}\omega{=}0\,|\hat{V}|\vec{q}\,'\omega\,'{=}0\rangle + \int \frac{d^3k}{(2\pi)^3}\,\frac{d\omega''}{(2\pi)}\; \langle q\omega{=}0\,|\bar{V}|\vec{q}\,''\omega''\rangle$$

$$\frac{i}{\frac{q''^2}{2m} - \frac{1}{2}E - \omega'' + i\varepsilon}\;\; \frac{i}{\frac{q''^2}{2m} - \frac{1}{2}E + \omega'' + i\varepsilon}\, \langle q\,''\omega''|\hat{v}|q\,'\omega\,'{=}0\rangle \;\; -$$

$$- \int \frac{d^3k''}{(2\pi)^3} \frac{d\omega''}{(2\pi)} \langle\vec{q}\omega{=}0|\hat{V}|\vec{q}\,''\omega''{=}0\rangle \frac{1}{\frac{q''^2}{m} - E + i\varepsilon} \langle\vec{q}\,''\omega''{=}0|v|\vec{q}\,'\omega{=}0\rangle \tag{18}$$

which involves the unknown quantity $\langle q''\omega''|\hat{T}|q'\omega'{=}0\rangle$ so that the integration in ω'' can not be carried out.

Of course to solve (14) is as difficult as to solve the original equation (11). If however the perturbative solution of (14) is good we can, by projecting it, get $\langle q\omega{=}0|v|q'\omega{=}0\rangle$.

In fact the perturbative solution of (14) is

$$v = \hat{V} + \hat{V}\,(G - g)\,\hat{V} + \hat{V}\,(G - g)\,\hat{V}\,(G - g)\,\hat{V} + \ldots \tag{19}$$

This can now be projected to give

$$\langle q\omega{=}0|v|q'\omega{=}0\rangle = \langle q\omega{=}0|\tilde{V}|q'\omega'{=}0\rangle + \int \frac{d^3q''}{(2\pi)^3} \frac{d\omega''}{(2\pi)} \langle q\omega{=}0|\hat{V}|q''\omega''\rangle \times$$

$$\times \left\{ \frac{i}{\frac{q''^2}{2m} - \frac{1}{2}E - \omega'' + i\varepsilon} \; \frac{i}{\frac{q''^2}{2m} - \frac{1}{2}E + \omega'' + i\varepsilon} \; \frac{1}{\frac{q''^2}{m} - E + i\varepsilon} \right\} \times$$

$$\times \; \langle q''\omega''|\hat{V}|q'\omega'{=}0\rangle + \ldots \tag{20}$$

and the integration in ω'' can be carried since $\langle q\omega{=}0|\hat{V}|q''\omega''\rangle$ is known and so is $\langle q''\omega''|\hat{V}|q'\omega'{=}0\rangle$. If the series converge it is then possible to get $\langle q\omega{=}0|v|q'\omega{=}0\rangle$ and solve the three-diniensional "relativistic" equation (16)

Finally one should note, again, that if $\langle\vec{q}\omega|\hat{V}|q'\omega'\rangle \; \langle\vec{q}|\hat{V}|\vec{q}\,'\rangle$ that is, if there is no dependence on relative energy, then equation (18) reduces immediately to

$$\langle\vec{q}\omega{=}0|v|\vec{q}\,'\omega{=}0\rangle = \langle\vec{q}|\hat{V}|\vec{q}\,'\rangle$$

that is, one regains once again, the Lippmann-Schwinger equation.

A question that immediately occurs is if it is possible to choose for g a form different from the one given by equation (15) that would make the series (20) to converge faster. The answer to that depends of course on the form of $<q\omega|\hat{V}|q'\omega'>$ but even given that, a general answer to this question is not known. Partial answers can be found in references (3) and (4).

4. THE THREE-BODY CASE

Correct integral equation for three particle scatteting problems, were given by Fadeev [13]. Those equations (3 coupled equations) replace for the three body case the Lippmann-Schwinger equation.

It is therefore interesting to inquire if it is possible to give relativistic generalizations of the Fadeev-equations, that is, to give three-dimensional Fadeev like equations which are consistent with Lorentz invariance. Many such a formulation [14] can be found in the literature, but those equations were not derived from the three particle Bethe-Salpeter equation. It is also known [15,16] that there are difficulties in carrying out this program. Those difficulties seen to be considerable reduced if one considers the problem in terms of the non-relativistic three particle Bethe-Salpeter equation. We hope to demonstrate this in a future paper.

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REFERENCES

1. F.A.B.Coutinho, American Journal of Physics, to appear.
2. A.A. Logunov and A.N. Tavkhelidze, Nuovo Cimento 29, 380 (1963).
3. R.M. Woloshin and A.D. Jackson, Nuclear Physics B64, 269 (1973).
4. V.A. Rizov ànd I.T. Todorov, Sov. J. of Part. Nuclei 6, 269 (1976).
5. R.H. Thompson, Phys.Rev. D1, 110 (1970).

6. A. Klein and T.H. Lee, Physical Review *D10*, 4380 (1974).
7. F. Gross, Phys. Rev. *186*, 1448 (1969).
8. C.Fronsdal and R.W.Huff, Phys.Rev. D3, 933 (1971).
9. M.M. Lévy, Phys.Rev. 88, 72 (1952); *88*, 725 (1952). W. Macke, Z. Naturforsch. *89*, 599 (1953). A. Klein, Phys.Rev. *90*, 1101 (1952).
10. J.G. Taylor, Phys.Rev. 250, 1321 (1966).
11. N. Nakamichi, Supplement of the Progress of Theoretical Physics, *43* (1969).
12. S. Machida, Progress of Theoretical Physics, *Supplement 39, 91* (1966).
13. L.D. Fadeev, *Mathematical Aspects of the Three-Body Problem in the Quantum Scattering Theory*. Israel Program for Scientific Translations, Jerusalen (1965).
14. R. Blankenbecler and R. Sugar, Phys.Rev. 242, 105 (1966). V. A. Alessandrini and R.L. Omnes, Phys. Rev. 139, B 167 (1965). R. Aaron, R. D. Amado and Y.Y. Yam, Phys. Rev. 140, B 1291 (1965).
15. V.M. Vinogradov, Teor.Mat.Fiz. *8*, 876 (1971).
16. W. Glöcke and R. Offermann, Ruhr Universität Bochun, Preprint RUB TP 11/157 (1976).