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Polarization of Electron and Photon Beams

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In this work the concepts of polarization of photons and relativistic electrons are introduced in a simplified form. Although there are important differences in the concepts of photon and electron polarization, it is shown that the same formalism may be used to describe them, which is very useful in the study of interactions concerned with polarizations of both photons and electrons, as in the photoelectric and Compton effects. Photons are considered, in this work, as a special case of relativistic spin 1 particles, with zero rest mass.

Neste trabalho são apresentados, de uma forma simplificada, os conceitos de polarização de fotons e eletrons. Embora existam importantes diferenças nos conceitos de polarização de fotons e eletrons, mostra-se que o mesmo formalismo pode ser usado para descreve-los, o que é muito útil no estudo de interações envolvendo polarizações de ambos, fotons e eletrons, como em efeitos fotoelétrico e Compton. Neste trabalho os fotons são considerados como um caso especial de partículas relativísticas de spin 1, com massa de repouso nula.

1. INTRODUCTION

During a research program concerned with correlations between polarizãtions of photons and electrons in relativistic photoelectric effect, it was noticed the lack of a text which brings together the concepts and simplified descriptions of polarization of electron and photon beams. That was the reason for trying to write such a text.

This work starts with the non relativistic description of polarized electrons and subsequently the relativistic description, when the relation between the electron beam polarization and the expectation value of the spin components of the electrons is not so obvious, is developed.

Following the description of spin 1/2 particles polarization, the formalism is extended for spin 1 particles and the photon polarization concept arises from the fact that photons are a special case of relativistic spin 1 particles with zero rest mass.

Although the concepts of electron and photon polarizations are quite different, it is shown that the same formalism can be applied to both cases.

2. NON RELATIVISTIC DESCRIPTION OF POLARIZATION OF OF ELECTRONS [1-6]

In this section some basic results from elementary non relativistic quantum mechanics will be presented in order to introduce the description of the electron polarization in terms of the spin of the electrons.

The observable spin, represented by the operator

$$\underline{s} = s_x \hat{e}_x + s_y \hat{e}_y + s_z \hat{e}_z \tag{2.1}$$

satisfies the commutation relations characteristic of angular momentum

$$[s_i, s_j] = i\hbar s_k \text{ (cyclic)} \tag{2.2}$$

and can be written as

$$\underline{s} = \frac{\hbar}{2} \underline{\sigma} \tag{2.3}$$

where σ_i are the Pauli matrices, which may be represented as

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2.4}$$

The wave function which describes the state of a particle of spin 1/2 is a two component spinor

$$\chi = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \tag{2.5}$$

and the most usual basis to represent this general state consists of the spinors

$$u_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad u_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{2.6}$$

which are eigenfunctions of a_z with eigenvalues +1 and -1, respectively.

Any general pure state χ can, therefore, be written as a linear combination of u_1 and u_2 (coherent states)

$$\chi = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = a_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + a_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{2.7}$$

This method of representing a pure state is called Jones formalism, due to the similarity with the method introduced by Jones[8] to describe polarized light.

If the state χ is normalised such as $\langle\chi|\chi\rangle = 1$, one gets the condition $|a_1|^2 + |a_2|^2 = 1$, where $|a_1|^2$ is the probability that the particle is in the state u_1, that is, the probability of finding the value $+\hbar/2$ whenmeasuringthespincomponent in z direction, and $|a_2|^2$ is the probability of finding $-\hbar/2$ (stateu,).

Due to the noncommutative relations (2.2) it is not possible to measure simultaneously the three components of the spin, which means physically that the measurement of one component affects the spin state. However, the operator $s^2 = s_x^2 + s_y^2 + s_z^2$ comutes with any component and can be measured simultaneously with any component. Its eigenvalues is $s(s+1)\hbar^2 = 3\hbar^2/4$. Therefore, when one says that the spin of the particle is in z direction, it means that the spin component in z direc-

tion has the value $\hbar/2$ but nothing can be said about the other components, only that $s_x^2 + s_y^2 = \frac{3}{4}\hbar^2 - \frac{1}{4}\hbar^2 = \frac{\hbar^2}{2}$.

So far only the spin of a single particle has been described. Consider now a system of electrons.

If all the electrons are in the same spin state, the system is said to be in a pure spin state $\chi = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$, and the spin direction is specified by a_1 and a_2.

The spinor that describes a spin in an arbitrary direction $\hat{e}$ is such that $(\underline{\sigma}.\hat{e})\,\chi = \lambda\chi$, where $\underline{\sigma}.\hat{e}$ is the projection of the spin operator in direction $\hat{e}$ and λ the correspoding eigenvalue. If one writes the general expression for $\hat{e}$ as

$$\hat{e} = \sin\theta\cos\phi\,\hat{e}_x + \sin\theta\sin\phi\,\hat{e}_y + \cos\theta\,\hat{e}_z \tag{2.8}$$

and substitutes the Pauli matrices as in (2.4) in the above expression, one gets the eigenvalues $\lambda_+ = +1$ and $\lambda_- = -1$ with the corresponding wave functions

$$\chi_+ = \begin{pmatrix} \cos\theta/2 \\ \sin\theta/2\,e^{i\phi} \end{pmatrix} \text{ and } \chi_- = \begin{pmatrix} \sin\theta/2\,e^{-i\phi} \\ -\cos\theta/2 \end{pmatrix} \tag{2.9}$$

Thus, χ_+ and χ_- are eigenfunctions of $\underline{\sigma}.\hat{e}$ with eigenvalues $+1$ and -1 and represent the states where the spin in the arbitrary direction $\hat{e}$ has the vaiues $+\hbar/2$ and $-\hbar/2$. The direction of the spin, $\hat{e}$, is determined by

$$\frac{a_2}{a_1} = \tan\frac{\theta}{2}\,e^{i\phi} \quad .$$

The importance of (2.9) is that it shows that it is possible to construct states which are simultaneously eigenstates of the momentum and spin component in any arbitrary direction since the solution of the Schrödinger equation for a free particle is $\psi = \chi \exp\frac{i}{\hbar}(p.r - Et)$. Relativistically, as will be shown in section (3), it is only possible to define such states in the direction of the momentum.

The eigenvalue representsthe result of a single measurement and if information about the average spin direction of a beam of elec-

trons is required, one has to calculate the expectation value of the Pauli operator, $\langle \underline{\sigma} \rangle$. The axial vector

$$\Lambda = \frac{\langle \underline{\sigma} \rangle}{I} \tag{2.10}$$

is called polarization of the beam, and I is the intensity of the beam and can be written as

$$I = \langle \sigma_0 \rangle = \chi^{\dagger}\chi \tag{2.11}$$

where σ_0 is the 2x2 unity matrix.

The four quantities $\langle \sigma_i \rangle$ $i = 0,1,2,3$ are called the Stokes parameters, first introduced for polarized light and give a complete description of the beam.

Using the representation (2.4) the Stokes parameters can be written as:

$$\begin{aligned} P_0 &= I = \langle \sigma_0 \rangle = |a_1|^2 + |a_2|^2 \\ P_1 &= P_z = \langle \sigma_z \rangle = |a_1|^2 - |a_2|^2 \\ P_2 &= P_x = \langle \sigma_x \rangle \quad a_1^* a_2 + a_1 a_2^* \\ P_3 &= P_y = \langle \sigma_y \rangle = i(a_1 a_2^* - a_1^* a_2) \end{aligned} \tag{2.12}$$

Therefore, from the definition of the Stokes parameters P_i = = $\langle \sigma_i \rangle$ and from (2.10) the relation between the polarization vector and the Stokes parameters is

$$\underline{\Lambda} = \frac{P}{P_0} \tag{2.13}$$

The degree of polarization is defined as

$$p = \sqrt{\Lambda_1^2 + \Lambda_2^2 + \Lambda_3^2} = \frac{\sqrt{P_1^2 + P_2^2 + P_3^2}}{P_0} \tag{2.14}$$

and has the value of the unity for a beam in which all the electrons are in the same spin state (totally polarized beam).

A simple representation of the beam is, thus, given bya four component column matrix

$$\begin{pmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{pmatrix} = I \begin{pmatrix} 1 \\ \Lambda_1 \\ \Lambda_2 \\ \Lambda_3 \end{pmatrix} \tag{2.15}$$

Four quantities are therefore required to describe the beam: its intensity and the three components of the polarization vector.

An alternative way of describing the beam is by introducing the density matrix defined as $\rho = |\chi\rangle\langle\chi|$,

$$\rho = \begin{pmatrix} |a_1|^2 & a_1 a_2^* \\ a_1^* a_2 & |a_2|^2 \end{pmatrix} \tag{2.16}$$

The density matrix, as any 2x2 matrix, can be expanded as a linear combination of the Pauli matrixes and the unity matrix. One property of the density matrix is that for any operator A, the expectation value can be found by the relation

$$\langle A \rangle = \frac{\text{trace } \langle \rho A \rangle}{\text{trace } \rho} \tag{2.17}$$

As $P_i = \langle \sigma_i \rangle$, one can write

$$P_i = \frac{\text{tr } \rho\sigma_i}{\text{tr } \rho} \text{ or } \underline{P} = \frac{\text{tr } \rho\underline{\sigma}}{\text{tr } \rho} \tag{2.18}$$

whichcanbeeasilyverified byusing (2.4), (2.12) and (2.16). Thus, from (2.12) and (2.16) one can represent the density matrix in terms of the Stokes parameters

$$\rho = \frac{1}{2} \begin{pmatrix} P_0 + P_1 & P_2 - iP_3 \\ P_2 + iP_3 & P_0 - P_1 \end{pmatrix} \tag{2.19}$$

which can be written as

$$\rho = \frac{1}{2} (P_0 + \underline{P} \cdot \underline{\sigma}) \tag{2.20}$$

or, in terms of the polarization vector,

$$\frac{\rho}{\mathrm{tr}\,\rho} = \frac{1}{2}\,(1 + \underline{\Lambda}\cdot\underline{\sigma}) \tag{2.21}$$

So far only totally polarized electrons have been considered, that is, ensembles in which all particles are in the same spin state. Electrons have two spin states in an arbitrary direction, and beams composed of electrons in these two states are represented by a density matrix which is the average of the individual matrices of the two states The two orthogonal states may be written as[9]

$$\chi_1 = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \qquad \chi_2 = \begin{pmatrix} a_2^* \\ -a_1^* \end{pmatrix} \tag{2.22}$$

and the average density matrix is

$$\bar{\rho} = \omega \begin{pmatrix} |a_1|^2 & a_1 a_2^* \\ a_1^* a_2 & |a_2|^2 \end{pmatrix} + (1-\omega) \begin{pmatrix} |a_2|^2 & -a_1 a_2^* \\ -a_1^* a_2 & |a_1|^2 \end{pmatrix} \tag{2.23}$$

where ω is the relative population of the state χ_1. The resultant beam is said to be partially polarized and the definitions of the Stokes parameters (2.18), (2.19) and (2.20) are still valid if the average density matrix is used. If the two states are equally populated, i.e. $\omega = 1/2$, the density matrix becomes $\rho = \frac{1}{2}\sigma_0$ and the beam is said to be unpolarized. A general partially polarized beam in the z direction is described by the Stokes vector

$$\begin{pmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{pmatrix} = \begin{pmatrix} |a_+|^2 + |a_-|^2 \\ |a_+|^2 - |a_-|^2 \\ 0 \\ 0 \end{pmatrix} \tag{2.24}$$

where $\dfrac{|a_+|^2}{|a_+|^2+|a_-|^2}$ is the probability that a measurement of the spin in

the z direction finds an eigenvalue $+\hbar/2$ and $\frac{|a_-|^2}{|a_+|^2+|a_-|^2}$ is the probability of finding eigenvalue $-\hbar/2$. The polarization of the beam is

$$\Lambda = \frac{P}{P_0} = \frac{N_+ - N_-}{N_+ + N_-} \tag{2.25}$$

where $N_+(N_-)$ is the number of measurements that yield tha value $+\hbar/2$ $(-\hbar/2)$, and $N_+ + N_-$ is the total nurnber of measurements. If N_+ or N_- is equal to zero the beam is totally polarized ($|\Lambda| = 1$), if $N_+ = N_-$ the beam is unpolarized (A = O), otherwise the beam is partially polarized ($0 < |\Lambda| < 1$).

If a beam is a mixture of individual subsystems with definite spin states in different directions, the polarization of the systern is defined as the average of the polarization values of the individual sybsystems which are in pure states $\chi^{(n)}$ [1],

$$\underline{\Lambda} = \frac{\sum_n P_0^{(n)} \underline{\Lambda}^{(n)}}{\sum_n P_0^{(n)}} \tag{2.26}$$

which can be rewritten, following (2.10) as

$$\underline{\Lambda} = \frac{\sum_n \langle\chi^{(n)} | \chi^{(n)}\rangle \; (\langle\chi^{(n)}|\sigma|\chi^{(n)}\rangle / \langle \chi^{(n)}|\chi^{(n)}\rangle)}{\sum_n \langle\chi^{(n)}|\chi^{(n)}\rangle},$$

which can be simplified to

$$\underline{\Lambda} = \frac{\sum_n \langle\chi^{(n)}|\sigma|\chi^{(n)}\rangle}{\sum_n P_0^{(n)}} \tag{2.27}$$

The density matrix of the ensemble is given by

$$\rho = \sum_n \rho^{(n)} \tag{2.28}$$

where $\rho^{(n)}$ are the individual density matrices of the pure states $\chi^{(n)}$ as defined in (2.16). The Stokes four vector is $\begin{pmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{pmatrix}$ where, now

$$\begin{aligned} P_0 &= \sum_n P_0^{(n)} = \sum_n |a_1^{(n)}|^2 + |a_2^{(n)}|^2 \\ P_1 &= \sum_n P_1^{(n)} = \sum_n |a_1^{(n)}|^2 - |a_2^{(n)}|^2 \end{aligned} \tag{2.29}$$

etc.

that is, the Stokes parameters of the total enseinbles are the sums of the individual Stokes parameters of each partial group of electrons. The relation (2.20) is still valid, butwith the redefinitions (2.28) and (2.29).

An example may illustrate the mixture of states in a beam. Consider an ensemble of N electrons totally polarized in the z direction mixed with another ensemble of N electrons totally polarized in the x direction. The resulting polarization is, from (2.27)

$$\underline{A} = \frac{N\,\hat{\Lambda}_z + N\,\hat{\Lambda}_x}{2N} = \frac{1}{2}\,\hat{\Lambda}_z + \frac{1}{2}\,\hat{\Lambda}_x$$

and the degree of polarization is $A = \sqrt{1/4 + 1/4} = \sqrt{1/2}$ and therefore the resultant beam is partially polarized. The density matrix is, from (2.28), (2.16), (2.4) and from $\chi^{(1)} = \sqrt{N}\,\binom{1}{0}$, $\chi^{(2)} = \sqrt{N/2}\,\binom{1}{1}$

$$\rho = N\left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\right] = N\begin{pmatrix} 3/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$$

The Stokes parameters of the total ensemble are

$$P_0 = 2N \qquad P_1 = 2N/2 \qquad P_2 = 2N/2 \qquad P_3 = 0$$

which are the sums of the individual Stokes parameters of the two subsystems.

To conclude the discussion about polarization of electrons it can be stated that an ensemble of electrons is said to be polarized if the electron spins have a preferential orientation so that there exists a direction for which the two possible spin states are not equally populated.

3. POLARIZATION OF RELATIVISTIC ELECTRONS[1,2,5,7,10,11]

The Schrödinger equation is not valid in relativistic quantum mechanics since it is not invariant under Lorentz transformation, and therefore a new description is required, which is obtained by the Dirac equation. The definition of polarization in terms of the expectation value of the spin is not Lorentz invariant and is valid only in the rest frame of the electrons. In this section the relativistic treatment of polarization of electrons will be discussed.

The Dirac equation says that $i\hbar \frac{\partial \psi}{\partial t} = H\psi$, where the Hamiltonian is given by

$$H = c\, \underline{\alpha}.\frac{\vec{\hbar}}{i} \underline{\nabla} + \beta\, m_0 c^2 \tag{3.1}$$

where $\underline{a}$ and β, 4×4 hermitian matrices, are called the Dirac matrices, and the wave function ψ is a 4 component column matrix. The four Dirac matrices satisfy the relations

$$\beta^2 = \alpha_i^2 = 1$$
$$\alpha_i\beta + \beta\alpha_i = 0 \text{ and } \alpha_i\alpha_k + \alpha_k\alpha_i = 2\,\delta ik \tag{3.2}$$

The usual representation of the Dirac matrices is

$$\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \underline{\alpha} = \begin{pmatrix} 0 & \underline{\sigma} \\ \underline{\sigma} & 0 \end{pmatrix} \tag{3.3}$$

where $\underline{\sigma}$, the Pauli matrices, are represented in (2.4).

A solution of the Dirac equation (3.1) for a free particle is

$$\psi(r,t) = u \exp\left[-\frac{ip_\nu x^\nu}{\hbar}\right] \tag{3.4}$$

where $p^\mu = (\frac{E}{c}, p)$, $x^\mu = (ct, r)$ and

u is a 4 component column matrix $u = \begin{matrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{matrix}$

From (3.1), (3.3), (2.4) and (3.4) one sees that the set of four equations contained in the Dirac equations gives non trivial solutions if the determinant

$$\begin{vmatrix} (E - m_0c^2)u_1 & 0 & -cp_z\, u_3 & -c(p_x - ip_y)u_4 \\ 0 & (E - m_0c^2)u_2 & -c(p_x + ip_y)u_3 & cp_z\, u_4 \\ -cp_z\, u_1 & -c(p_x - ip_y)u_2 & (E + m_0c^2)u_3 & 0 \\ -c(p_x + ip_y)u_1 & cp_z\, u_2 & 0 & (E + m_0c^2)u_4 \end{vmatrix}$$

vanishes. The condition for non trivial solutions, therefore, leads to the energy relation

$$E^2 = (p_x^2 + p_y^2 + p_z^2)c^2 + m_0^2c^4$$

$$E = \pm\sqrt{p^2c^2 + m_0^2c^2} \tag{3.5}$$

There are Four linearly independent solutions, two belongimg to the positive energy, E_+ (which corresponds to electrons) and two belonging to the negative energy, E_- (which corresponds to positrons). The solutions are, after normalization:

$$u_A = \sqrt{\frac{|E|+m_0c^2}{2|E|}} \begin{pmatrix} 1 \\ 0 \\ cp_z/(E_+ + m_0c^2) \\ c(p_x + ip_y)/(E_+ + m_0c^2) \end{pmatrix}$$

$$u_B = \sqrt{\frac{|E|+m_0c^2}{2|E|}} \begin{pmatrix} 0 \\ 1 \\ c(p_x - ip_y)/(E_+ + m_0c^2) \\ -cp_z/(E_+ + m_0c^2) \end{pmatrix} \qquad (3.6)$$

$$u_C = \sqrt{\frac{|E|+m_0c^2}{2|E|}} \begin{pmatrix} -cp_z/(-E_- + m_0c^2) \\ -c(p_x + ip_y)/(-E_- + m_0c^2) \\ 1 \\ 0 \end{pmatrix}$$

$$u_D = \sqrt{\frac{|E|+m_0c^2}{2|E|}} \begin{pmatrix} -c(p_x - ip_y)/(-E_- + m_0c^2) \\ cp_z/(-E_- + m_0c^2) \\ 0 \\ 1 \end{pmatrix}$$

Solutions u_A and u_B correspond to positive energy (electrons) and u_C and u_D to negative energy (positrons). The general solution for positive energy is a linear combination of u_A and u_B

$$\psi = (a_1u_A + a_2u_B)\, e^{i(\underline{k}.\underline{r}-\omega t)} \qquad (3.7)$$

where $u = a_1u_A + a_2u_B$, that is,

$$u = \sqrt{\frac{E+m_0c^2}{2E}} \left\{ a_1 \begin{pmatrix} 1 \\ 0 \\ cp_z/(E+m_0c^2) \\ c(p_x+ip_y)/(E+m_0c^2) \end{pmatrix} + a_2 \begin{pmatrix} 0 \\ 1 \\ c(p_x-ip_y)/(E+m_0c^2) \\ -cp_z/(E+m_0c^2) \end{pmatrix} \right\} \qquad (3.8)$$

Each solution has two components, u_3 and u_4 which, in the non relativistic limit are of order of v/c which is small. These components are called the small components and the others (u_1 and u_2) are the large components, which in non relativistic limit correspond to the solution of the Schrödinger equation.

When one calculates the time variation of the angular momentum $\underline{L}$ of a free particle using the relation

$$\frac{d\underline{L}}{dt} = \frac{1}{i\hbar}\left[\underline{L},H\right] \quad \text{one gets}$$

$$\frac{d\underline{L}}{dt} = c\,\underline{\alpha} \times \underline{p}$$

which does not agree with the classical and non relativistic quantum mechanics results where, for a free particle (or particle under a central potential) the orbital angular momentum is a constan: of motion. The operator which, added to $\underline{L}$, commutes with H is

$$\underline{S} = \frac{\hbar}{2}\,\underline{\sigma}' \qquad (3.9)$$

where

$$\sigma' = \begin{pmatrix} \sigma & 0 \\ 0 & \sigma \end{pmatrix} \qquad (3.10)$$

and σ are the non relativistic Pauli matrices. The operator $\underline{S}$ is the spin operator of the particles and has eigenvalues $\pm \hbar/2$, which means that a particle obeying the Dirac equation has spin 1/2. Therefore, one can see that the spin appears directly from the Dirac equation. Although this derivation was made for a specific representation of the Dirac matrices, it is valid for any representation.

In the non relativistic case, it is possible to construct eigenfunctions of $\underline{\sigma}.\hat{\underline{e}}$, with eigenvalues ± 1, for any arbitrary direction $\hat{\underline{e}}$ by the coherent super-position of eigenstates of σ_z, as shown in (2.9). However, in the relativistic case, that is no longer possible. Consider, for example, the operator σ'_z, applied to the general solution of the Dirac equation (3.8),

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ \frac{c}{E+m_0c^2}\,(a_1 p_z + a_2(p_x - ip_y)) \\ \frac{c}{E+m\,c^2}\,(a_1(p_x+ip_y) + a_2 p_z) \end{pmatrix} = \begin{pmatrix} a_1 \\ -a_2 \\ \frac{c}{E+m_0c^2}\,(a_1 p_z + a_2(p_x - ip_y)) \\ \frac{-c}{E+m_0c^2}\,(a,\,(p_x+ip_y) - a_2 p_z) \end{pmatrix} \tag{3.11}$$

There is no possible combination of values of a_1 and a_2 which makes the state u an eigenstate of σ'_z with eigenvalue +1 or -1, unless $p_x = p_Y = 0$. If the operator σ'_x is applied to the wave function u, the condition such that this function can be eigenstate of σ'_x is, analogously, $p_y = p_z = 0$. Therefore, it is only possible to construct states which are simultaneously eigenstates of the momentum and the spin component along the momentum direction. This result is expected because in the relativistic treatment it is not the spin which is a constant of motion, but the total angulat momentum $\underline{L} + \underline{S}$. Only if $\underline{L} = 0$ is the spin constant. For a plane wave in the arbitrary $\hat{\underline{e}}$ direction, the component of the orbital momentum in this direction, L_e, vanishes and therefore it is posssible to find eigenstates of S_e.

It can therefore be seen that for relativistic electrons, only in their rest frame, where $(p_x)_R = (p_y)_R = (p_z)_R = 0$, is it possible to find eigenvalues of the spin component operators, in any arbitrary direction. The spin part of the eigenfunction (3.8), in the rest frame is $u_R = \begin{pmatrix} a_1 \\ a_2 \\ 0 \\ 0 \end{pmatrix}$, which means that only the large components are not equal to zero. For low energy electrons the small components are so small that it is possible to say that one can find eigenstates of any spin component.

As the non relativistic Pauli rnatrices used in the definition of the spin operator (3.9) and (3.10) are in a representation where σ_z is diagonal, it is convenient to choose the z axis as the direction of the momentum, because some simplification will arise, as will be shown. If $\underline{p} = p_z \hat{z}$, the general solution, for electrons, of the Dirac equation, represented in (3.7) becomes

$$\psi = u\, e^{i(kz-\omega t)} = \sqrt{\frac{E+m_0c^2}{2E}} \begin{pmatrix} a_1 \\ a_2 \\ a_1 cp/(E+m_0c^2) \\ -a_2 cp/(E+m_0c^2) \end{pmatrix} e^{i(kz-\omega t)} \qquad (3.12)$$

The small components can be expressed in terms of the large cornponents,

$$u_3 = \frac{cp}{E+m_0c^2}\, u_1 = A\, u_1 \quad \text{and} \quad u_4 = \frac{-cp}{E+m_0c^2}\, u_2 = -A\, u_2 \qquad (3.13)$$

and therefore Jones formalism, used for non relativistic electrons, may be applied, but the difference is that now u_1 and u_2 are the large components of the Dirac spinor.

From (3.7) one sees that there are two independent solutions u_A and u_B for electrons of momentum $\boldsymbol{p}$, corresponding to orthogonal spin states with spin parallel and antiparallel to the direction of the momentum.

An arbitrary pure state may be written as $|\chi> = a\,\chi_+> + b|\chi_->$ and the density matrix can be constructed.

$$\rho = |\chi><\chi| = \begin{pmatrix} |a|^2 & ab^* \\ a^*b & |b|^2 \end{pmatrix} \qquad (3.14)$$

in the same form as (2.16). As any 2×2 matrix, the density matrix canbe written as a linear combination of the Pauli rnatrices,

$$\rho = \frac{1}{2}\,(P_0 + \underline{P}.\underline{\sigma})$$

which has the same form of (2.20), and as before $(P_0, \underline{P})$ are defined as

the Stokes parameters. One has now to interpret the relation between the Stokes parameters and the polarization vector with the expectation value of the spin for relativistic electrons. In the electrons rest frame the former quantities are still related to the expectation value of the spin operator as

$$(P_i)_R = (\langle \sigma_i' \rangle) \quad , \quad (\underline{\Lambda})_R = \left(\frac{P}{P_0}\right)_R .$$

However, in the laboratory frame the three vector polarization does not have the same form, since it is not a Lorentz covariant quantity. The relation between the Stokes parameters (and, consequently, the polarization vector) and the expectation values of the components of the spin operator are understood in a simples way if one takes the convenient choice of taking the z axis as the direction of momentum and uses the representation (2.4), (3.3) and (3.10). In this case, as already pointed out, the solutions of the Dirac equation are of the form of (3.12) and the quantities a and b in (3.14) may be identified as the large components a_1 and a_2 of the Dirãc spinor. The expectation values of the components of the spin operator are written as

$$\langle \sigma_i' \rangle = \langle u | \sigma_i' | u \rangle \quad i = 0,1,2,3$$

where $|u\rangle$ is given by (3.12) and σ_i' is written in the representations (3.10), (2.4). Therefore, one has

$$\langle \sigma_0' \rangle = \frac{(E+m_0c^2)}{2E} (a_1^* \; a_2^* \; a_1^* A - a_2^* A) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_1 A \\ -a_2 A \end{pmatrix}$$

$$\langle \sigma_0' \rangle = \frac{1}{(1 + A^2)} (1 + A^2)(|a_1|^2 + |a_2|^2) = |a_1|^2 + |a_2|^2 = P_0 \qquad (3.15)$$

since $\dfrac{2E}{E \; c \; m_0c^2} = 1 + A^2$.

$$\langle\sigma_1'\rangle = \frac{1}{1+A^2}\,(a_1^*\,a_2^*\,a_1^*\,A - a_2^*\,A)\begin{pmatrix}1&0&0&0\\0&-1&0&0\\0&0&1&0\\0&0&0&-1\end{pmatrix}\begin{pmatrix}a_1\\a_2\\a_1A\\-a_2A\end{pmatrix}$$

$$\langle\sigma_1'\rangle = \frac{1}{(1+A^2)}\,(|a_1|^2 - |a_2|^2)(1+A^2) = |a_1|^2 - |a_2|^2 = P_1 \tag{3.16}$$

$$\langle\sigma_2'\rangle = \frac{1}{(1+A^2)}\,(a\quad a_2^*\,a_1^*\,A - a_2^*\,A)\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}\begin{pmatrix}a_1\\a\\a_1A\\-a_2A\end{pmatrix}$$

$$\langle\sigma_2'\rangle = \frac{1}{(1+A^2)}\,(a_1\,a_2^* + a_1^*\,a)\,(1-A^2)$$

$$\langle\sigma_2'\rangle = \frac{(1-A^2)}{(1+A^2)}\,P_2 = \frac{m_0c^2}{E}\,P_2 \tag{3.17}$$

Similarly

$$\langle\sigma_3'\rangle = \frac{m_0c^2}{E}\,P_3 \tag{3.18}$$

Relations (3.16), (3.17) and (3.18) show that the longitudinal polarization, $\Lambda_1 = P_1/P_0$, keeps the non relativistic meaning, because

$$\langle S_1\rangle = \frac{1}{2}\,\hbar\,P_1$$

However, the transverse polarization components do not have the same rneaning as expectation value of the corresponding spin components because

$$\langle S_2\rangle = \frac{m_0c^2}{E}\,\frac{1}{2}\,\hbar\,P_1 \qquad \langle S_3\rangle = \frac{m_0c^2}{E}\,\frac{1}{2}\,\hbar\,P_2 \quad .$$

Therefore in the rest frame the expectation values of the spin components have the same meaning as the polarization components whereas for low energy electrons, which have $E \sim m_0c^2$, the same inter-

pretation is still a reasonable approximation. However, as the energy of the electrons increases, the transverse components of the spin tend to zero whereas the polarization components remain finite. In the extreme relativistic limit, the spin is longitudinal.

4. SPIN 1 PARTICLES [5,6,12,13]

The formalism used in section (2) for non relativistic spin 1/2 particles will now be extended for spin 1 particles. The reason for studying spin 1 particles is that, as will be shown, photons are a special case of relativistic spin 1 particles with mass zero.

For non relativistic particles with spin 1, the spin operator $\underline{S}$ satisfied the same commutation relations (2.2)

$$\begin{aligned} [S_i, S_j] &= i\hbar S_k \\ [S^2, S_i] &= 0 \end{aligned} \tag{4.1}$$

where now the operators S_i, S^2 are 3x3 matrices and

$$S^2 = S_x^2 + S_y^2 + S_z^2 = 2\hbar^2 1_3 \tag{4.2}$$

where 1_3 is the unit 3x3 matrix.

There are three possible states of the components of the spin along any direction, corresponding to eigenvalues $m = +\hbar, 0, -\hbar$.

In a representation where S_z is diagonal, one can write

$$S_z = \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{4.3}$$

and the eigenstates of S_z are the three cornponent spinors

$$\chi_+ = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad \chi_0 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad \chi_- = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{4.4}$$

corresponding to eigenvalues $+\hbar$, 0, $-\hbar$, respectively.

Any general pure state of the particle can be written in terms of the orthogonal bases formed by χ_+, χ_0 and χ_- as

$$\chi = a_+\chi_+ + a_0\chi_0 + a_-\chi_- \tag{4.5}$$

If this general state represents one particle the normalization condition is $|a_+|^2 + |a_0|^2 + |a_-|^2 = 1$ and if it represents a totally polarized beam, the same quantity is normalized to be the intensity of the beam.

Through the commutation relations (4.1) and (4.2) the matrices corresponding to S_X and S_Y are found to be, in this representation

$$S_x = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad S_y = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix} \tag{4.6}$$

The density matrix $\rho = \chi\chi^\dagger$ can be written as

$$\rho = \begin{pmatrix} |a_+|^2 & a_+a_0^* & a_+a_-^* \\ a_+^*a_0 & |a_0|^2 & a_0a_-^* \\ a_+^*a_- & a_-a_0^* & |a_-|^2 \end{pmatrix} \tag{4.7}$$

In analogy with the Pauli matrices for spin 1/2 particles one can define

$$\underline{S} = \hbar\,\underline{\sigma} \tag{4.8}$$

but it is no longer possible to represent any 3×3 matrix in terms of 1_3, σ_x, σ_y, σ_z. In order to give a complete representation of a general 3×3 matrix for spin 1 particles, nine quantities must be specified and the conventional representations uses the following set of spherical tensor operators[5,12]

$$\tau_{00} = 1_3$$

$$\tau_{1\pm1} = \mp \frac{\sqrt{3}}{2} (\sigma_x \pm i\sigma_y) \quad \tau_{10} = \frac{\sqrt{3}}{2} \sigma_z$$

$$\tau_{2\pm2} = \frac{\sqrt{3}}{2} (\sigma_x \pm i\sigma_y)^2 \quad \tau_{2\pm1} = \frac{-\sqrt{3}}{2} \left[(\sigma_x \pm i\sigma_y)\sigma_z + \sigma_z(\sigma_x \pm i\sigma_y) \right] \tag{4.9}$$

$$\tau_{20} = \frac{\sqrt{2}}{2} (3\sigma_z - 21_3)$$

The expectation values of the tensor operators τ_{ij} are

$$t_{ij} = \langle \tau_{ij} \rangle = \frac{\text{tr } (\rho \tau_{ij})}{\text{tr } \rho} \tag{4.10}$$

Substituting (4.6) and (4.9) in (4.10), the density matrix can be written as

$$\rho = \frac{1}{3} t_{00} \begin{vmatrix} 1 + \sqrt{\frac{3}{2}}\, t_{1,} & + \sqrt{\frac{1}{2}}\, t_{20} \sqrt{\frac{3}{2}}\, t_{1-1} + \sqrt{\frac{3}{2}}\, t_{2-1} & \sqrt{3}\, t_{2-2} \\ -\sqrt{\frac{3}{2}}\, t_{11} - \sqrt{\frac{3}{2}}\, t_{21} & 1 - \sqrt{2}\, t_{20} & \sqrt{\frac{3}{2}}\, t_{1-1} - \sqrt{\frac{3}{2}}\, t_{2-1} \\ \sqrt{3}\, t_{22} & -\sqrt{\frac{3}{2}}\, t_{11} + \sqrt{\frac{3}{2}}\, t_{21} & 1 - \sqrt{\frac{3}{2}}\, t_{10} + \sqrt{\frac{1}{2}}\, t_{20} \end{vmatrix} \tag{4.11}$$

The nine quantities which describe the beam may be divided into three parts: one component of a tensor of rank zero, t_{00}, which is the intensity of the beam; three components of a tensor of rank 1, t_{1i}, which form the so-called vector polarization; five components of a tensor of rank 2, t_{2i}, which form the tensor polarization. Unlike the case of spin 1/2 particles, the intensity and vector polarization do not describe completely the beam, because the components of the 2nd rank tensor are also observables.

The expectation values of the tensor operator τ_{ij} may be written, in terms of the matrix elements of the density matrix as:

$$t_{00} = I = |a_+|^2 + |a_0|^2 + |a_-|^2$$

$$t_{10} = (|a_+|^2 - |a_-|^2)/I$$

$$t_{1\pm1} = \frac{1}{\sqrt{2}} (a_\pm^* a_0 + a_\mp a_0^*)/I \qquad (4.12)$$

$$t_{20} = \frac{1}{\sqrt{2}} (|a_+|^2 - 2|a_0|^2 + |a_-|^2)/I$$

$$t_{2\pm1} = \sqrt{3/2}\ (a_\pm a_0^* - a_\pm^* a_0)/I$$

$$t_{2\pm2} = \sqrt{3}\ (a_\pm^* a_\mp)/I$$

As for spin 1/2 particles, one can define the Stokes parameters for spin 1 particles, but now, instead of four, one has to define nine components,

$$S^\dagger = I(1\ t_{11}\ \ t_1\ t_{1-1}\ t_{22}\ \ t_{21}\ \ t_{2-1}\ t_{2-2}) \qquad (4.13)$$

and the meaning of each component is not so clear.

As in the case of spin 1/2 particles, the Schrödinger equation is not valid for relativistic spin 1 particles, which must be described by a Lorentz covariant equation. In the case of spin 1/2 particles this equation is the Dirac equation (3.1). Particles with restmass m_0 and spin zero are described by a scalar field $\psi(x)$ which satisfies the Klein-Gordon equation[14]

$$\left(\Box - \frac{m_0 c^2}{\hbar^2} \right) \psi(x) = 0 \qquad (4.14)$$

where $\Box = \partial_\mu \partial_\mu$, $\mu = 0,1,2,3$.

Particles with spin 1 could possibly be described by a four vector field which transforms under Lorentz transformation as $\psi_\mu(x') = a_{\mu\nu}\ \psi_\nu(x)$ ans whose components ψ_μ satisfy the Klein-Gordon equation

$$\left(\Box - \frac{m_0 c^2}{\hbar^2} \right) \psi_\mu(x) = 0 \qquad \mu = 0,1,2,3 \qquad (4.15)$$

However, the four vector representation is not a unique spin representation since the four dimensional representation $\mathcal{D}^{\frac{1}{2}}\mathcal{D}^{\frac{1}{2}}$ of the proper Lorentz group, which is derived from the direct product $\mathcal{D}^{\frac{1}{2}\,\frac{1}{2}} \approx \mathcal{D}^{\frac{1}{2}} \times \mathcal{D}^{\frac{1}{2}}$ of $\mathcal{D}^{\frac{1}{2}}$ representation of the three dimensional rotational group, is reduced to the direct sum of different spatial rotation representations, that is,

$$\mathcal{D}^{\frac{1}{2}\,\frac{1}{2}} \approx \mathcal{D}^{\frac{1}{2}} \times \mathcal{D}^{\frac{1}{2}} = \mathcal{D}^{1} + \mathcal{D}^{0} \tag{4.16}$$

where $\mathcal{D}^{0}$ is the scalar representation associated with spin zero and $\mathcal{D}^{1}$ is the vector representation associated with spin 1. As the representation space of $\mathcal{D}^{1}$ is three dimensional, there must be one subsidiary condition which limits the number of independent field components to three, in order to cut out the scalar representation. This condition, which must also be a Lorentz covariant, is called be Lorentz condition and can be written as[14]

$$\partial_{\mu}\psi_{\mu} = 0 \tag{4.17}$$

The set of equations (4. 15) and (4.17) describes particles with rest mass m_0 and a unique spin 1 and is called Proca equations.

A special case of particles satisfying the Proca equations is that of zero rest mass particles. Such particles are the photons, which are the quanto of electromagnetic radiation. If one substitutes the rest mass in equation (4.15) with zero, the equations becomes

$$\Box\psi_{\mu} = 0 \tag{4.18}$$

which is the well known wave equation. The polarization states of the photons will be studied in the next section.

As in the case of relativistic spin 1/2 particles, it is not possible to construct eigenstates of relativistic spin 1 particles with a definite momentum in an arbitrary direction. There are, however, simultaneous eigenstates of the momentum and the spin component along the momentum direction since the component of the orbital angular momentum along the direction of momentum vanishes. Once more it is convenient to choose the z axis as the direction of momentum, and therefore a gene-

ral pure state of a relativistic spin 1 particle can be written as a linear combination of the eigenstates of the spin component along the direction of motion, corresponding to eignevalues $+\hbar$, 0, $-\hbar$. Equations (4.4), (4.5), (4.7) are still valid, with the condition that they are related with the spin component along the direction of motion.

5. POLARIZATION OF PHOTONS [3,5,6,13,14]

The electromagnetic field $A_\mu = (i\phi, \underline{A})$, as mentioned in section (4) is a special case of the fields which describe spin 1 particles, with the particular property that the quanta associated with such a field, the photons, have a zero rest mass. The Proca equations (4.15) and (4.17) for the electromagnetic field become

$$\begin{aligned} \nabla^2 \underline{A} - \frac{1}{c^2} \frac{\partial^2 A}{\partial t^2} &= 0 \\ \nabla^2 \phi - \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} &= 0 \end{aligned} \tag{5.1}$$

with the Lorentz condition

$$\nabla . \underline{A} + \frac{1}{c} \frac{\partial \phi}{\partial t} = 0 \tag{5.2}$$

It is important to notice that while the wave equations (5.1) are a particular case of the more general Proca equations, the Lorentz condition (5.2) is the same as for finite rest mass particles.

The Lorentz condition (5.2) does not exhaust the possible gauge transformations on the electrornagnetic field A_μ and a very useful gauge is the so called Coulomb or transverse gauge which says that

$$\nabla . \underline{A} = 0 \tag{5.3}$$

In the Coulomb gauge, from (5.2) and the second equation (5.3) one can see that the scalar potential ϕ vanishes. This means that for free electromagnetic waves it is possible to choose a gauge in which the

scalar potential vanishes and the electromagnetic fiels is totally described by the poteritial vector $\underline{A}$ satisfying the wave equation

$$\Box \underline{A} = 0 \tag{5.4}$$

subjected to the condition *(5.3)*. The solution of *(5.4)* is a plane wave

$$\underline{A}(r,t) = \underline{\hat{e}}\, e^{\frac{i}{\hbar}(\underline{p}.\underline{r} - \frac{E}{c}t)} \tag{5.5}$$

and the condition *(5.3)* says that

$$\underline{\hat{e}}.\underline{\hat{p}} = 0 \tag{5.6}$$

that is, the electromagnetic waves are transverse. For this reason condition *(5.3)* is called the transversality condition.

It is convenient to choose the z axis as the direction of propagation of the field because in this case any vector operator can be written as

$$\underline{A}(x,y,z) = A_+(\hat{e}_+,\hat{e}_0,\hat{e}_-)\chi_+ + A_0(\hat{e}_+,\hat{e}_0,\hat{e}_-)\chi_0 + A_-(\hat{e}_+,\hat{e}_0,\hat{e}_-)\chi_- \tag{5.7}$$

where

$$\chi_+ = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \chi_0 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \chi_- = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{5.8}$$

are eigenstates of the spin component along the z axis with eigenvalues $+\hbar$, 0, $-\hbar$ respectively, and

$$\hat{e}_+ = \frac{-1}{\sqrt{2}}(\hat{e}_x + i\hat{e}_y) \qquad \hat{e}_0 = \hat{e}_z \qquad \hat{e}_- = \frac{1}{\sqrt{2}}(\hat{e}_x - i\hat{e}_y) \tag{5.9}$$

form the spherical basis which diagonalises S_z, which is then represented by *(4.3)* and form an irreducible tensor of rank one which transforms under rotation in the same way as the spherical harmonics Y_{11}, Y_{10} and $Y_{1,-1}$, respectively. In this representation the transversality

condition (5.6) for the electromagnetic field allows a simplified form of the vector potential to be written as

$$\underline{A} = A_{+}\chi_{+} + A_{-}\chi_{-} \text{ , since } A_0 = 0 \qquad (5.10)$$

where χ_+ and χ_- are the eigenvectors of S_z with eigenvalues $+\hbar$ and $-\hbar$. This representation is called the helicity representation and the state χ_+ is called the state of positive helicity or state of right circular polarization (RCP), corresponding to photons with spin along the direction of momentum. χ_- is the state of negative helicity, or left circular polarization (LCP), corresponding to photons with spin opposite to the direction of momentum. The helicity, defined as the compcnentofthe spin in the direction of motion (S_z in this case) is a constant of motion since the component of the orbital angular momentum along the direction of momentum is zero, that is $J_z = S_z$.

The transverse character of the electromagnetic field which is a consequence of the vanishing rest mass of the photons can therefore be interpreted to correspond to a zero probability of finding a photon with spin component $m_s = 0$ along the direction of motion, which means that photons are longitudinally polarized and have only two possible spin states, the right and left circular polarized states. In analogy with the study of spin 1/2 particles one can say that as the velocity of the particle increases the transverse components of the spin of the particle decreases and in the extreme relativistic case whete $v=c$, which is the case of the photons, the state corresponding to transverse spin vanishes and the particle has only longitudinal spin component. Of course, as the photons have a rest mass equal to zero, it is meaningless to try to define the spin of the photon in its rest frame.

The massless character of the photon, which corresponds to $a_0 = 0$, leaves only four independent non vanishing elements of the density matrix (4.7), and consequently only four independent non vanishing expectation values of the tensor operators $\tau_{..}$ defined in (4.12). These elements are $t_{00} = |a_+|^2 + |a_-|^2$ which is the intensity of the beam,

$$t_{10} = \sqrt{\frac{3}{2}}\,(|a_+|^2 - |a_-|^2) \qquad (5.11)$$

which is a measure of the degree of circular polarization and $t_{2\pm2} = \sqrt{3}\, a_{\pm}^{*} a_{\pm}$ which are associated with the degree of linear polarization.

Since there are only two possible spin states for the photons, one can apply the same formalisrn used for non relativistic electrons to describe the polarization of photons, but a reinterpretation of the parameters is necessary. A general pure state may be written as

$$\chi = a_{+}\chi_{+} + a_{-}\chi_{-} = a_{+}\begin{pmatrix}1\\0\end{pmatrix} + a_{-}\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}a_{+}\\a_{-}\end{pmatrix} \tag{5.12}$$

where χ_{+} and χ_{-} are two component spinors representing states of RCP and LCP, and $|a_{+}|^2$ and $|a_{-}|^2$ are the probabilities of finding a photon in such states.

Alternatively, the transverse condition also allows a representation of the vector potential as

$$\underline{A} = A_x\, \hat{e}_x + A_y\, \hat{e}_y \tag{5.13}$$

and one can represent a general pure state as

$$\chi = a_x\, \chi_x + a_y\, \chi_y = a_x\begin{pmatrix}1\\0\end{pmatrix} + a_y\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}a_x\\a_y\end{pmatrix} \tag{5.14}$$

where χ_x and χ_Y are states corresponding to linear polarization along two mutually orthogonal axes, x and y, perpendicular to the direction of momentum, and $|a_x|^2$, $|a_y|^2$ are the probabilities of finding a photon in such states. This formalism is known as Jones formalism, and was first introduced in the analysis of polarized light[8]. The state χ in (5.14) is called the Jones vector and all the formalism is analogous to the coherent states used for non relativistic electrons, although the Jones vector is a two component vector in real space while thefunctions describing the electrons are in spin space (spinors).

One can define the density matrix for a beam of photons in the same pure state exactly as (2.16), and for a mixed beam as (2.28). Also, the density matrix may be written in terms of the Stokes parameters as in (2.19), (2.20) although the Stokes parameters and Pauli matrices, in the case of photons, are not defined in real space but in

Poincaré space. The photon beam is, thus, completely described by the four Stokes parameters:

$$P_0 = I = \langle \sigma_0 \rangle = |a_x|^2 + |a_y|^2 \tag{5.15}$$

the intensity of the beam is the total number of photons;

$$P_1 = \langle \sigma_1 \rangle = |a_x|^2 - |a_y|^2 \tag{5.16}$$

the difference between the number of photons polarized along the x and y axes, ir; a measure of the linear polarization along the two transverse axes x and y ;

$$P_2 = \langle \sigma_2 \rangle = a_x\, a_y^* + a_x^*\, a_y \tag{5.17}$$

is a rneasure of the linear polarization along axes rotated by $\pi/4$ in relation to the axes of P_1 ;

$$P_3 = \langle \sigma_3 \rangle = i(a_x\, a_y^* - a_x^*\, a_y) \tag{5.18}$$

is a measure of the circular polarization.

The polarization is defined as

$$\Lambda_i = P_i/P_0 \qquad i = 1,2,3 \tag{5.19}$$

but the polarization "vector" is not defined in real space, but in the Poincaré space in which the Stokes pararneters are defined.

States in the helicity and Jones representation defined in (5.11) and (5.13) can be related by using the relations (5.9) between the basis vectors. In Jones representations, the RCP and LCP states are represented by $\binom{1}{i}$ and $\binom{1}{-i}$ and by substituting the relations between a_+, a_- with a_x, a_Y one gets

$t_{00} \doteq |a_x|^2 + |a_y|^2 = P_0$ (intensity of the beam)

$t_{10} \doteq i(a_x a_y^* - a_x^* a_y) = P_3$ (degree of circular polarization)

$t_{2\pm2} \doteq (|a_x|^2 - |a_y|^2 \pm i(a_x a_y^* + a_x^* a_y)) = P_1 \pm iP_2$ (combinations of linear polarization) which is in agreernent with che interpretation of (5.11). ($\doteq$ means equal except for constants).

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