Revista Brasileira de Física, Vol. 11, Nº 4, 1981

Spinor Monopole Harmonics and the Pauli Spin Equation

J. G. PEREIRA[+] and P. LEAL FERREIRA[+]

Instituto** de **Física** Teórica, **São Paulo

Recebido em 28 de Fevereiro de 1982

In the framework of Wu and Yang theory of $U(1)$ magnetic monopoles, two problems are revisited in this work: (i) the binding of a spin-0 monopole to a spin-1/2 particle possessing an arbitrary magnetic dipole moment, and (ii) the energy levels and properties of the electron-dyon system. In both problems, the spin-1/2 particle is assumed to obey the Pauli spin equation. Spin-orbit and other higher order terms are treated as a perturbation, in connection with the second mentioned problem. Wu and Yang's spinor monopole harmonics allow an elegant and simplified treatment of those problems. The results obtained are in good agreement with those obtained in older papers.

No contexto da teoria de Wu e Yang dos monopolo-magnéticos abelianos, dois problemas são revisitados neste trabalho: (i) A ligação de um monopolo magnético de spin zero a uma partícula de spin 1/2, dotada de um momento de dipolo magnético arbitrário, e (ii) os níveis de energia e outras propriedades do sistema elétron-dyon. Em ambos os problemas a particula de spin 1/2 é tratada pela equação de Pauli. O acoplamento spin-órbita e outros termos superiores são tratados perturbativamente para o segundo problema mencionado. Os harmônicos espinoriais de Wu e Yang permitem um tratamento elegante e simplificado daqueles problemas. Os resultados estão em bom acordo com os obtidos em trabalhos anteriores.

+ With a Fellowship by FAPESP, São Paulo

++ With support of FINEP, Rio de Janeiro

1. INTRODUCTION

A few years ago, Wu and Yang[1] propounded a new formulation for the system consisting of a charged particle and a U(1) magnetic monopole, in which Dirac's string singularities in the vector potential[2] are completely absent.

Their theory, from the mathematical view point, has the geometrical structure of a fiber bundle. In other words, the conciliation of eletromagnetism with magnetic rnonopoles and quantum mechanics, leads naturally to a nontrivial U(1) fiber bundle[3]. As a consequence, instead of wave functions, one arrives at the concept of wave-sections. In particular, the eigen-sections of the mornentum operator for a spinless charged particle in the magnetic field of a monopole, are the so called monopole harmonics $Y_{q,\ell,m}(\Omega)$, which are generalizations of the ordinary spherical harmonics. The additional label q - denoting the product of the electric charge of the particle times the magnetic charge of the monopole - is an integer or half-integer that specifies how the wave-sections, defined in two overlapping regions R_a and R_b around the monopole, are related in the region of overlap.

The ccncept of monopole harmonics and their generaiizations provides an important simplification in the treatment of problems involving magnetic monopoles, as compared with older treatments based on symmetric top wave functions $d^J_{m,n}(\theta)$. We hope that this will becorne apparent in the following sections, where two problems are revisited and solved by means of Wu and Yang's method. The first one, to be discussed in section 2, is that of the binding of a spinless monopole to a spin-1/2 particle possessing an arbitrary magnetic dipole mornent, a problem first attacked by kalkus[4] and later by Sivers[5], in connection with the possibility of binding of a magnetic monopole to a spin-1/2 atomic nucleus. In this problem, the spin-1/2 particle is treated by means of the Pauli spin equation, neglecting the spin-orbit interaction and higher relativistic terms. Clearly, the applications of these results to the monopole-nucleus system is, in several aspects, an admittedly rough procedure serving, at most, as order of magnitude estimates. The inadequacy of the treatment and small distances can be partly circumvented by a cut-off introduced by means of a hard repulsive core with a radius equal to the nuclear radius, as first suggested by Sivers[5].

The second problem, to be discussed in section 3, is that of the energy spectrum of the electron-dyon quantum system, with its respective degeneracy pattern. Also studied is the electric dipole moment of the same system. In our treatrnent, the electron is described by a Pauli spin equation, the spin orbit and higher order terms being considered as a perturbation.

In both problems, the relevant harmonics are the two- components spinor monopole harmonics, discussed by Kazama, Yang and Goldhaber[6] in relation with their treatment of the Dirac equation[7] for a charged particle in the field of a magnetic monopole. A small Appendix contains the relevant results on those harmonics that will be used in this work. Finally, section 4 is devoted to the main conclusions.

2. BINDING OF A SPINLESS U(1) MAGNETIC MONOPOLE TO A CHARGED PARTICLE OF SPIN-1/2 WITH A GIVEN MAGNETIC MOMENT B_Z

Following refs. 4 and 5, we treat the above problem in the Pauli spin approximation ($\hbar$ = c = 1)

$$H\psi \equiv \left[\frac{1}{2T}(\vec{p} - Z|e|\vec{A})^2 - B_Z \frac{|e|\nu g}{2M_1} \frac{\vec{\sigma}.\vec{r}}{r^3} + V(r)\right]\psi = E\psi \tag{2.1}$$

where T is the reduced mass of the system, νg is the magnetic monopole charge $(\nu = \pm 1, \pm 2, \ldots)$, $Z|e|$ is the electric charge of the particle, and B_Z is its number of nuclear magnetons $[|e|/2M_1$, where M_1 is the proton mass]. Possible relativistic corrections are included in $V(r)$. In order to avoid string singularities in the vector potential, $\vec{A}$ is defined[1] as two functions $(\vec{A})_a$ and $(\vec{A})_b$ in two overlapping regions R_a and R_b arciund the monopole. Consequently, ψ is a section and in order to solve Pauli equation (2.1), one has simply to consider the total angular momentum

$$\vec{J} = \vec{L} + \vec{\sigma}/2 \tag{2.2}$$

which is an Hermitian operator in the Hilbert space of sections. In (2.2), $\vec{L}$ is defined as

$$\vec{L} = \vec{r} \times (\vec{p} - Z|e|\vec{A}) - q\frac{\vec{r}}{r} \qquad (2.3)$$

where $q = Z|e|\nu g$, and $\vec{\sigma}$ are the Pauli spin matrices. The eigen-sections of $\vec{J}^2$ and J_Z are the spinor monopole harmonics, whose main properties are reproduced in the Appendix.

Setting

$$\vec{P} = \vec{p} - Z|e|\vec{A} \qquad (2.4)$$

we have. the identity

$$\frac{1}{2T}(\vec{\sigma}.\vec{P})^2 \equiv \frac{1}{2T}(\vec{p} - Z|e|\vec{A})^2 - \frac{Z|e|\nu g}{2T}\frac{\vec{\sigma}\cdot\vec{r}}{r^3} \qquad (2.5)$$

so that, the Pauli equation (2.1) can be written as

$$\left[\frac{1}{2T}(\vec{\sigma}.\vec{P})^2 + \frac{\nu}{2}\left(\frac{Z}{T} - \frac{B_Z}{M_1}\right)|e|g\frac{\vec{\sigma}.\vec{r}}{r^3} + V(r)\right]\psi = E\psi \qquad (2.6)$$

This equation will be solved, separately, for two cases. In the first one, we will study the state with angular momentum $j = |q| - 1/2$; in the second, the states with $j \geqslant |q| + 1/2$. Writting the angular momentum in the form

$$j = |q| - 1/2 + N\ , \qquad (2.7)$$

then the state with $j = |q| - 1/2$ correspond to take N=0, and the states with $j \geqslant |q| - 1/2$ correspond to take N integer $\geqslant 1$ in (2.7).

2.A. The lowest angular momentum state

The lowest angular momentum state (N=0) has

$$j = |q| - 1/2 \qquad (2.8)$$

and is described by the two component spinor (see Appendix)

$$\eta_m = \phi^{(2)}_{qjm}, \quad j = |q| - 1/2 \tag{2.9}$$

Making use of properties P.5 and P.6 of the Appendix in the spin equation (2.6) with the "Ansatz"

$$\psi_m = f(r)\eta_m \tag{2.10}$$

we readily obtain the following radial equation

$$-\frac{1}{r^2}\frac{d}{dr}\left[r^2\frac{d}{dr}\right]f(r) + \frac{\beta_0}{r^2}f(r) + 2T(E-V)f(r) = 0 \tag{2.11}$$

where

$$\beta_0 = |e|\nu g\left(Z - \frac{T}{M_1}B_Z\right)\frac{q}{|q|} \tag{2.12}$$

By Dirac's quantization condition[2]

$$|e|g = \frac{1}{2} \quad \text{or} \quad 2q = Z\nu \quad , \tag{2.13}$$

and equation (2.12) becomes

$$\beta_0 = \frac{|\nu|}{2}\left(Z - \frac{T}{M_1}B_Z\right) . \tag{2.14}$$

Note that: β_0 may be negative (attraction), depending on the values of the various quantities involved. In particular, for the proton ($Z = 1$ and $B_1 = 2{,}79$), β_0 is always negative for $M_g > 0{,}56\ M_1$, and positive for $M_g < 0{,}56\ M_1$. On the other hand, for instance, for the He^3 nucleus ($Z = 2$ and $B_2 = -2{,}12$), β_0 is always positive. Table I gives typical values of β_0 for He^3 and for the proton with various assumptions about the magnetic monopole charge and mass.

It may be remarked that for the neutron case ($Z=0$), the lowest angular momentum state is absent, so that for this case, the possible N values are

$$N = 1, 2, 3, ..$$

which will be studied in section 2.B.

Table I. Typical values of β_0, given by expression (2.14).

Nucleus	B_Z	ν	M_g ($M_1=1$)	β_0
p, H ($Z=1$)	2,79	±1	1,16 100 200	-0,25 -0,88 -0,89
		±2	0,81 100 200	-0,25 -1,77 -1,78
He^3	-2,12	±1	100 200	4,09 4,13
		±2	100 200	8,17 8,27
C^{13} ($Z=6$)	0,702	±1	32,2 100 200	-0,25 -1,04 -1,28
		±2	23,3 100 200	-0,25 -2,09 -2,57
F^{19} ($Z=9$)	2,63	±1	4,46 100 200	-0,25 -16,6 -18,3
		±2	4,32 100 200	-0,25 -33,1 -36,6

The results obtained in this section coincide with those of Sivers, except that the sign of ν is to assume, correctly, either value, positive or negative.

2.B. Higher angular momentum states

For angular momentum states with

$$j \geq |q| - 1/2 \quad \text{or} \quad N \geq 1 \ , \tag{2.15}$$

we take the "Ansatz"

$$\psi_{jm} = f(r) \left[\xi_{jm}^{(1)} + K\xi_{jm}^{(2)}\right] \tag{2.16}$$

in terms of the spinar monopole harmonics $\xi_{jm}^{(i)}$, defined in the Appendix. Using, now, the properties P.7 to P.10 in equation (2.6) with the wave section (2.16), we obtain two 2nd order differential equations for $f(r)$

$$\frac{1}{r^2}\frac{d}{dr}\left[r^2 \frac{d}{dr}\right]f - \frac{\mu(\mu-1) + \chi K}{r^2} f + 2T(E-V)f = 0 \tag{2.17}$$

$$\frac{1}{r^2}\frac{d}{dr}\left[r^2 \frac{d}{dr}\right]f - \frac{\mu(\mu+1) + \chi/K}{r^2} f + 2T(E-V)f = 0 \tag{2.18}$$

where

$$\mu = \left[N(N + |Z\nu|)\right]^{1/2} \tag{2.19}$$

and

$$\chi = \frac{\nu}{2} \left[\frac{T}{M_1} B_Z - Z\right] \tag{2.20}$$

Eqs. (2.17) and (2.18) are made to coincide into a single differential equation if

$$\mu(\mu-1) + \chi K = \mu(\mu+1) + \frac{\chi}{K} \ . \tag{2.21}$$

Solving for K, one gets

$$K^{(1)} = \frac{\mu-(\mu^2 + \chi^2)^{1/2}}{\chi} \tag{2.22}$$

and

$$K^{(2)} = \frac{\mu+(\mu^2 + \chi^2)^{1/2}}{\chi} \tag{2.23}$$

Then, the two possibles wave sections are given by

$$\psi^{(1)}_{jm} = f^{(1)}\left[\xi^{(1)}_{jm} + K^{(1)}\ \xi^{(2)}_{jm}\right] \tag{2.24}$$

and

$$\psi^{(2)}_{jm} = f^{(2)}\ \left[\xi^{(1)}_{jm} + K^{(2)}\ \xi^{(2)}_{jm}\right] , \tag{2.25}$$

where the function $f^{(1,2)}$ is obtained as the solution of

$$\frac{1}{r^2}\frac{d}{dr}\left[r^2\frac{d}{dr}\right]f^{(1,2)} - \frac{\beta_N^{(1,2)}}{r^2}f^{(1,2)} + 2T(E-V)f^{(1,2)}=0 \tag{2.26}$$

with

$$\beta_N^{(1,2)} = \mu^2 \mp (\mu^2+\chi^2)^{1/2} \tag{2.27}$$

By using, now, (2.19) and (2.20), we finally obtain

$$\beta_N^{(1,2)} = N(N+|Z\nu|) \mp \left[N(N+|Z\nu|) + \frac{\nu^2}{4}\left(\frac{T}{M_1}B_Z - Z\right)^2\right]^{1/2} , \tag{2.28}$$

where the up (down) sign corresponds to the solution $\psi^{(1)}$ ($\psi^{(2)}$). Again, as in the lowest angular momentum case, this number may be positive or negative, depending on the values of the various quantities involved. In particular, for the proton, with $M_g \gtrsim M_1$ (where M_g is the monopole mass) $\beta_N^{(1,2)}$ is always positive, in contrast with β_0 for the N=O case. Table II givestypical values of $\beta_N^{(1,2)}$ for He^3 and proton with various assumptions about the magnetic monopole charge and for differents angular momentum states.

2.C. The radial equation

The radial equation is

$$\frac{1}{r^2}\frac{d}{dr}\left[r^2\frac{d}{dr}\right]f - \frac{\beta}{r^2}f + 2T(E-V)f = 0 \tag{2.29}$$

Table II: Typical values of $\beta_N^{(1,2)}$, given by exprersion (2.28) for several spin 1/2 nuclei

Nucleus	B_Z	M_g $(M_1=1)$	ν	N	$\beta_N^{(1)}$	$\beta_N^{(2)}$
n (Z = 0)	-1,91	200	±1	1	-0,38	2,38
				2	1,79	6,21
				3	5,85	12,15
			±2	1	-1,15	3,15
				2	1,24	6,76
				3	5,45	12,55
p, H (Z = 1)	2,79	200	±1	1	0,33	3,67
				2	3,39	8,61
				3	3,42	15,58
			±2	1	0,52	5,48
				2	4,66	11,34
				3	10,74	19,26
He^3 (Z = 2)	-2,12	200	±1	1	-1,48	7,48
				2	2,99	13,01
				3	9,34	20,66
			±2	1	-3,56	15,56
				2	3,04	20,96
				3	11,55	30,45
C^{13} (Z = 6)	0,702	200	±1	1	3,25	10,75
				2	11,20	20,80
				3	21,17	32,83
			±2	1	6,59	19,41
				2	20,51	35,49
				3	36,45	53,55
F^{19} (Z = 9)	2,63	200	±1	1	5,51	14,49
				2	16,33	27,67
				3	29,20	42,80
			±2	1	11,27	26,73
				2	31,01	48,99
				3	52,81	73,19

with 8 given by β_0 for $j = |q| - 1/2$ $(N=0)$ and by β_N for $j\ 2\ |q| + 1/2$ $(N\ 2\ 1)$. As pointed out in sections 2.A and 2.8, β may be positive or negative. In either case, the above equation may give rise to bound--states, as first pointed out by Sivers[5].

For completeness, we briefly discuss the bound state solutions for the case of an electrically uncharged monopole. In this case, the Coulomb potential vanishes in *(2.29)*. Then if $\beta>0$, no bound states, of course, exist. If $\beta<0$, the particle falls in the center, where the monopole is: there is no lower bound for the energy E. However, if $V(r)$ represents an infinite repulsive hard core at some small distance r,, that is, if

$$V(r) = \begin{cases} \infty & \text{for} \quad 0 < r < r_0 \\ 0 & \text{for} \quad r > r, \end{cases} \tag{2.30}$$

then, equation *(2.29)* may give rise to definite bound states. This situation is physically reasonable if the particle is a spin 1/2 atomic nucleus interacting with the monopole. At very short distances, the hadronic interactions may then be simulated by a potential like *(2.30)* at distances r, corresponding to the nuclear radius $(r_0 = 1.2\ A^{1/3}F\ )$. As first shown in ref. 5, one can then get binding energies much larger than those predicted by Malkus[4].

The boundary conditions to be imposed to the radial equation

$$\frac{d}{dr}\left[r^2\frac{d}{dr}\right]f - \left[k^2r^2 + S(S+1)\right]f = 0 \tag{2.31}$$

where $k^2 = -2TE$, are now

$$\begin{aligned} f(r_0) &= 0 \\ \lim_{r\to\infty} f(r) &= 0\ . \end{aligned} \tag{2.32}$$

The corresponding negative energy solution is given by[5]

$$f(r) = r^{-1/2}\,K_p(kr)\,, \tag{2.33}$$

where K_p is the modified Bessel function of order

$$p = \left(\beta + \frac{1}{4}\right)^{1/2} \tag{2.34}$$

The modified Bessel functions has no zeros, unless p be purely imaginary, that is

$$\beta < -\frac{1}{4} \tag{2.35}$$

In this case, approximate values for the energy, are easily seen to be given by

$$E_0 \simeq \frac{\beta + 1/4}{2Tr_0^2} \tag{2.36}$$

Notice the sensitivity of E_0 to the value of r_0. This type of binding occurs only for systems with $\beta<-1/4$. This is the case of the protonfor $N=0$ and $M_g \gtrsim M_1$, as can be seen from table I. On the other hand, for He^3 and neutron this type of binding can occur only in some cases, depending on the values of M_g, ν and of the angular momentum N (see table II). Typical values of the brinding energy *(2.36)* for several nuclei, are given in table III.

Table III: Typical values of the binding energy, given by equation (2.36), with $r_0 = bA^{1/3}$ ($b = 1{,}2F$ except for n and p where the value $b = 0{,}8$ L was taken).

Nucleus	M_g ($M_1 = 1$)	ν	r_0 [F]	N	β_N	E_0 [MeV]
n	200	±1	0,8	1	-0,38	4,24
		±2	0,8	1	-1,15	29,4
p	200	±1	0,8	0	-0,89	20,9
		±2	0,8	0	-1,78	49,9
He^3	200	±1	1,7	1	-1,48	2,99
		±2	1,7	1	-3,56	8,05
C^{13}	200	±1	2,8	0	-1,28	0,22
		±2	2,8	0	-2,57	0,50
F^{19}	200	±1	3,2	0	-18,3	2,11
		±2	3,2	0	-36,6	4,25

3. THE ELECTRON-DYON QUANTUM SYSTEM

We treat in this section the bound states of the electron--dyon quantum system in the non-relativistic approximation and in two differents cases. In either case, the dyon is considered as a spinless particle with magnetic charge νg ($\nu = \pm 1, \pm 2, \ldots$), electric charge λe ($\lambda = \pm 1, \pm 2, \ldots$) and infinite mass. The two cases are:

(i) we neglect the electron spin, describing it by Schrödinger equation, and

(ii) we take into account the electron spin and use the Pauli equation to describe it.

(i) The Schrödinger Case

The Schrödinger equation is ($\hbar = c = 1$)

$$H\psi \left[\frac{1}{2M}(\vec{p} - e\vec{A})^2 + \frac{\lambda e^2}{r}\right]\psi = E\psi \tag{3.1}$$

where M stands for the electron mass. Putting

$$\psi = R(r)\ Y_{q\ell m}(\Omega) \tag{3.2}$$

where $Y_{q\ell m}(\Omega)$

$$q = e\nu g\ , \tag{3.3}$$

one readily gets for the radial equation

$$\frac{1}{2M}\left[-\frac{d^2}{dr^2} - \frac{2}{r}\frac{d}{dr} + \frac{\ell(\ell+1) - q^2}{r^2}\right]R + \frac{\lambda e^2}{r}\ R = ER \tag{3.4}$$

where the range of R is given by

$$\ell = |q|,\ |q| + 1,\ |q| + 2, \ldots\ . \tag{3.5}$$

Notice that the equation (3.4) coincides with the usual radial equation for the H-atom, if an angular momentum number S is defined as

$$S = \left[(\ell + \tfrac{1}{2})^2 - q^2\right]^{1/2} - q^2 . \qquad (3.6)$$

Setting $k^2 = -2ME$, $M\lambda e^2/k = \gamma$ and $\rho = 2kr$, one gets

$$\frac{d^2R}{d\rho^2} + \frac{2}{\rho}\frac{dR}{d\rho} - \left[\frac{1}{4} - \frac{\gamma}{\rho} + \frac{S(S+1)}{\rho^2}\right] R = 0 , \qquad (3.7)$$

whose bound state solutions are given by

$$R(\rho) = e^{-\rho/2} \rho^S \, {}_1F_1 \, (-\gamma + S + 1, 2S + 2; \rho) . \qquad (3.8)$$

In order to guarantee the correct behavior at $\rho\to\infty$, one must have

$$-\gamma + S + 1 \equiv - n \qquad (3.9)$$

with n a positive integer. In this case, the confluent hypergeometric function ${}_1F_1$ appearing in (3.8), is a polynomial of degree $n = \gamma - S - 1$, and the energy spectrum is given by

$$E = - \frac{1}{2} Me^4 \gamma^{-2} = - \frac{1}{2} Me^4 (n + S + 1)^{-2} \qquad (3.10)$$

which is Balmer-like[8,9] and is represented in Fig.1 for $\nu=1$. Notice that there is no accidental degeneracy in the present case. The normalization of the wave function (3.8) may be performed using the well-known integrals involving confluent hypergeometric functions[10]. Finally, we remark that the wave functions of the ground state is zero at the origin, in contrast with the H-atom case.

(ii) The Pauli Spin Case

The electron-dyon system in the present case, corresponds to a particular case of the problem studied in section 2, with

$$B_Z = Z = 1 \qquad (3.11)$$

where B_Z, now, is expressed in Bohr magnetons. Then, the Pauli equation now reads

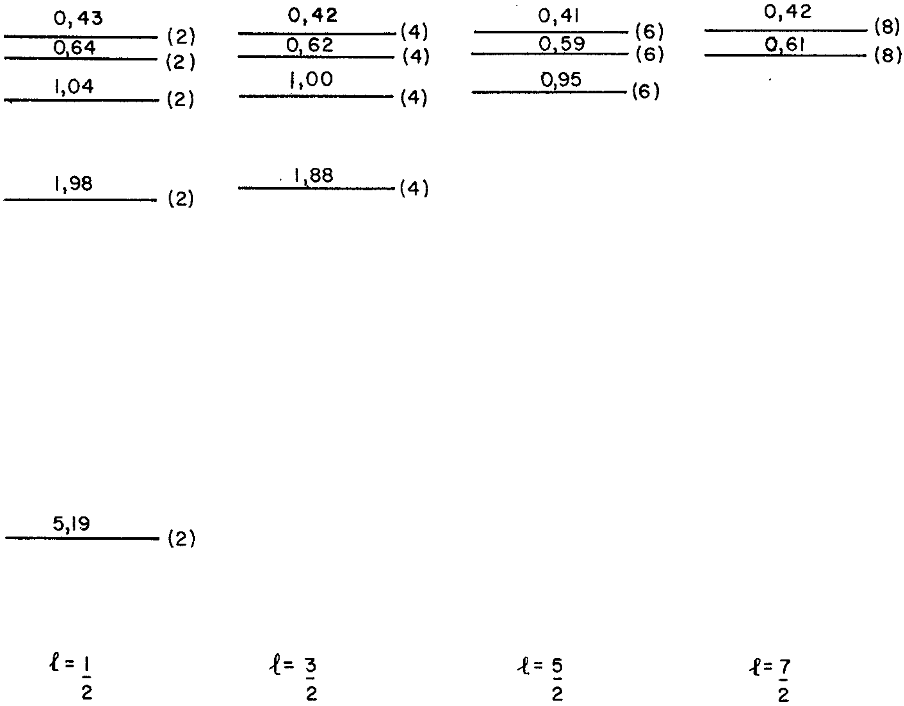

Fig.1 - Spectrum corresponding to the Schrödinger case, with $|\nu| = 1$. The degeneracy of all states is given by $2\ell+1$. Are also given the energy in eV for each state, according to expression (3.10).

$$H\psi \equiv \left\{\frac{1}{2M}\,[\vec{\sigma}(\vec{p} - e\vec{A})]^2 - \frac{\lambda e^2}{r}\right\}\psi = E\psi \,. \tag{3.12}$$

Again, we treat first the states with $j = |q| - 1/2$ (or $N=0$), whose wave sections are given by

$$\psi_m = f(r)\,\eta_m \quad , \tag{3.13}$$

with the spinor monopole harmonics η_m defined by (A.10). Using property P.6 of the Appendix and putting $\rho = 2(-2ME)^{1/2}r$ ($E<0$), we obtain the radial equation

$$\frac{1}{\rho^2}\,\frac{d}{d\rho}\left[\rho^2\,\frac{d}{d\rho}\right]f - \left[\frac{1}{4} - \frac{\gamma}{\rho}\right]f = 0 \tag{3.14}$$

where $\gamma = \lambda e^2(-2ME)^{1/2}/2E$. However, this equation **does** not have any solution that fulfills the boundary condition

$$f(0) = 0 \tag{3.15}$$

This occurs because the Hamiltonian corresponding to (3.12), is not a properly defined operator for treating wave sections with angular dependente of the type of that given by (3.13). The reason for this is contained in the discussion of Lipkin, Weisberger and Peshkin[11], who pointed out that the Jacobi identity is not satisfied for the components of $\vec{\Gamma} = \vec{p} - e\vec{A}$, that is

$$[[P_1, P_2], P_3] + [[P_2, P_3], P_1] + [[P_3, P_1], P_2] = -4\pi q\, \delta^3(\vec{r}) \ . \tag{3.16}$$

For the Schrödinger case, the Lipkin Weisberger and Peshkin difficulty does not appear, since all wave functions vanish at the origin. However, this does not occur in the present case and, to remedy the situation, we must[12] provide the electron with an "extra" magnetic moment so that the total magnetic moment, in Bohr magnetons, is given by

$$1 + \kappa \tag{3.17}$$

where κ is taken to be infinitesimal. With this assumption, the Pauli equation (3.12), now, becomes

$$H\psi \equiv \left\{\frac{1}{2M}\,[\vec{\sigma}(\vec{p} - e\vec{A})]^2 + \kappa\,\frac{\nu}{2M}\,\frac{\vec{\sigma}.\vec{r}}{r^3} - \frac{\lambda e^2}{r}\right\}\psi = E\psi, \tag{3.18}$$

and the radial equation (3.14), in turn, is

$$\frac{1}{\rho^2}\,\frac{1}{d\rho}\left[\rho^2\frac{d}{d\rho}\right]f - \left[\frac{1}{4} - \frac{\gamma}{\rho} + \frac{\beta_0}{\rho^2}\right]f = 0 \ , \tag{3.19}$$

with β_0 an infinitesimal given by

$$\beta_0 = \frac{|\nu|}{2}\,\kappa \ , \tag{3.20}$$

where we made use of the Dirac quantization condition

$$2|q| = |\nu| \ . \tag{3.21}$$

The solution of this equation that fulfills the boundary condition (3.15), is

$$f(\rho) = e^{-\rho/2} \rho^{S} {}_1F_1 \; (-\gamma + S + 1, \; 2S + 2; \; \rho) \tag{3.22}$$

where

$$S = (\beta_0 + \tfrac{1}{4})^{1/2} - \tfrac{1}{2} \tag{3.23}$$

is also an infinitesimal. The correct behavior at $\rho\to\infty$, requires that

$$-\gamma + S + 1 \simeq - \gamma + 1 = -n \tag{3.24}$$

where n is a positive integer (n = 0,1,2,...). In this case ${}_1F_1$ is a polynomial of degree n, and we have

$$E = - \frac{M\lambda^2 e^4}{2(n+1)^2} \quad . \tag{3.25}$$

We remark that these energies are independent of q and identical with the Balmer energies. In figure 2, where the energy spectrum of the Pauli spin case is represented, those energies correspond to the N = 0 tower.

The states with $j \geq |q| + 1/2$ are discussed next. In this case, we have $\chi = 0$ (see eq. 2.20), and then the two possible wave sections, which now are K-independent (see eqs. 2.17 and 2.18), are given by

$$\psi^{(1)}_{jm} = {}^{(1)} \xi^{(1)}_{jm} \qquad \text{and} \qquad \psi^{(2)}_{jm} = f^{(2)} \xi^{(2)}_{jm} \quad . \tag{3.26}$$

Using the properties P.7 to P.10 in equation (3.19) with the wave sections above, and putting $\rho \equiv 2(-ME)^{1/2}$ and $\gamma = \lambda e(-2ME)^{1/2}/2E$, we obtain two 2nd order differential equations

$$\frac{1}{\rho^2}\frac{d}{d\rho}\left[\rho^2 \frac{d}{d\rho}\right] f^{(i)} - \left[\frac{1}{4} - \frac{\gamma}{\rho} + \frac{\beta_N^{(i)}}{\rho^2}\right] f^{(i)} = 0; \; i = 1,2 \tag{3.27}$$

with

$$\mu = [N(N+|\nu|)]^{1/2} \quad , \tag{3.28}$$

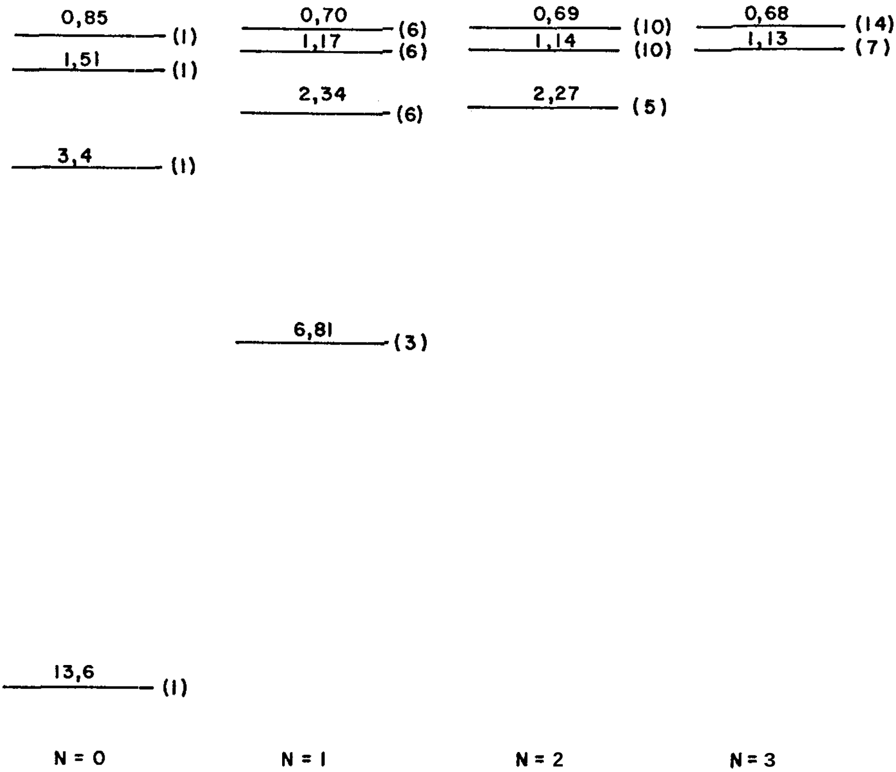

Fig.2 - Spectrum corresponding to the Pauli case, with $|\nu| = 1$. The lowest state in a tower is $2N+1$ degenerate; the others are $2(2N+1)$. Also given are the energy in eV for each state, according to expression (3.30) and (3.31).

$$\beta_N^{(i)} = \begin{cases} \mu(\mu-1) = N(N+|\nu|) - [N(N+|\nu|)]^{1/2} & \text{for } i = 1 \\ \\ \mu(\mu+1) = N(N+|\nu|) + [N(N+|\nu|)]^{1/2} & \text{for } i = 2\,, \end{cases} \tag{3.29}$$

and where we take the infinitesimal extra magnetic moment κ to be zero, since for the states under consideration, the radial wave sections vanish at the origin. The corresponding eigensolutions are

$$f^{(1)}(\rho) = \rho^{\mu-1}\, e^{-\rho/2}\; {}_1F_1(-\gamma + \mu,\ 2\mu;\ \rho) \tag{3.30}$$

with

$$E^{(1)} = -\frac{M\lambda^2 e^4}{2(n_1+\mu)}\ ;\ n_1 = \gamma-\mu\ ;\ n_1 = 0,\ 1,\ 2,\ \dots$$

and

$$f^{(2)}(\rho) = \rho^{\mu} e^{-\rho/2} {}_1F_1(-\gamma + \mu + 1, 2\mu + 2; \rho)$$

with (3.31)

$$E^{(2)} = -\frac{M\lambda^2 e^4}{2(n_2+\mu+1)^2} ; n_2 = \gamma-\mu-1; n_2 = 0,1,2, \ldots$$

Then it is clear that, for a given j-value, with the exception of the $n=0$, all the other states show a double degeneracy, besides the degeneracydue to theangularmomentum, given by $2j+1$. Using (2.7), and taking into account the Dirac quantization condition (3.21), one obtains $2j+1 = 2N+|\nu|$. The complete spectrum with the respective degeneracy, for the case with $|\nu|=1$, is given in Fig. 2. Notice that there is no accidental degeneracy, as is the case in the H-atom.

Let us now compute the fine structure corrections for the states with $j\geq|q| + 1/2$, which are those that have double degeneracy, for $\nu=1$. These corrections can be obtained by using perturbation theory, and considering as unperturbed states those solutions to the Pauli spin equation (3.18) with $\kappa=0$. The perturbation operator is obtained[13] by making the non-relativistic limit of the Dirac equation, and taking only the second order terms in $(1/c)$. So, we obtain

$$V = -\frac{1}{2M}\left[E + \frac{\lambda\alpha}{r} - \frac{e}{2M}\vec{\sigma}.\vec{B}\right]^2 + \frac{\lambda\alpha}{4M^2r^3}\left[\vec{\sigma}.\vec{L} + q\,\frac{\vec{\sigma}.\vec{r}}{r}\right] + \frac{\lambda\alpha\pi}{2M^2}\,\delta(\vec{r}) \qquad (3.32)$$

where E is the energy, $\vec{B}$ is the magnetic field of the dyon, $\vec{L}$ is the orbital angular momentum operator, given by

$$\vec{L} = \vec{r} \times (\vec{p}-e\vec{A}) - q\,\frac{\vec{r}}{r}, \qquad (3.33)$$

and

$$\alpha = e^2 \simeq 1/137 .$$

For the states with $j\geq|q| + 1/2$, as already discussed, there are two possible wave sections, given by (3.26). Then, the fine structure corrections can be obtained, for the two cases, by computing the mean value of the perturbation operator

$$\Delta E^{(i)} = <\psi_{jm}^{(i)}|V|\psi_{jm}^{(i)}>; \quad i = 1,2 . \tag{3.34}$$

Using the properties P.7 and P.8, and the orthogonality of the eigen-sections $\xi^{(i)}$'s, we obtain

$$\Delta E^{(i)} = -\frac{1}{2M}\left[E^{(i)2} + \lambda^2\alpha^2 <r^{-2}> + 2E^{(i)}\lambda\alpha <r^{-1}> + \frac{(e\nu g)^2}{4M^2} <r^{-4}>\right] +$$

$$+ \frac{\lambda\alpha}{4M^2} <\vec{\sigma}.\vec{L}> <r^{-3}> \tag{3.35}$$

where $E^{(i)}$ is given by (3.30) for $i=1$, and by (3.31) for $i=2$. The last term of (3.32) does not appear, since the $\phi^{(i)}$ vanish at the origin. The mean values $<r^{-n}>$, can be calculated making use of the well known integrals involving confluent hypergeometric functions[10]. The final results are

$$\Delta E^{(1)} = -\frac{M(\lambda\alpha)^4}{2(n_1+\mu)^3}\left[\frac{1}{(\mu-\frac{1}{2})} - \frac{3}{4(n_1+\mu)} + \frac{3}{(2\mu-3)(2\mu-2)(2\mu-1)(2\mu)(2\mu+1)} +\right.$$

$$\left. - \frac{1}{2(n_1+\mu)(2\mu-3)(4\mu^2-1)} + \frac{c^2(j-\frac{1}{2}) + s^2(j+\frac{3}{2})}{\mu(2\mu-1)(\mu-1)}\right] \tag{3.36}$$

and

$$\Delta E^{(2)} = -\frac{M(\lambda\alpha)^4}{2(n_2+\mu+1)^3}\left[\frac{1}{(\mu+\frac{1}{2})} - \frac{3}{4(n_2+\mu+1)} + \frac{3}{(2\mu-1)(2\mu)(2\mu+1)(2\mu+2)(2\mu+3)} +\right.$$

$$\left. - \frac{1}{2(n_2+\mu+1)^2(2\mu+3)(4\mu^2-1)} - \frac{s^2(j-\frac{1}{2}) - c^2(j+\frac{3}{2})}{\mu(\mu+1)(2\mu+1)}\right] \tag{3.37}$$

where c , s and μ are given, respectively, by (A.13), (A.14) and (3.28), and use was made of the Dirac quantization condition (3.21) with $\nu=1$. These corrections eliminate the double degeneracy that was present for all states with $N\geq 1$ and $n_2 + 1 = n$ ■ 1 (see fig 2), giving rise to an energy splitting, given by

$$\Delta E = \Delta E^{(2)} - \Delta E^{(1)} \tag{3.38}$$

By substituting $\Delta E^{(2)}$ and $\Delta E^{(1)}$, eqs. (3.36) and (3.27), and expressing c, s and j in terms of μ (eq. (3.28)), we get

$$A'' = \frac{M(Aa)^4}{2(n_2+\mu+1)^{\hat{}}} \left[\frac{32\mu^4 - 104\mu^2 + 87}{2(4\mu^2-1)(\mu^2-1)(4\mu^2-9)} - \frac{3}{(n_2+\mu+1)^2(4\mu^2-1)(4\mu^2-9)} \right.$$

$$\left. - \frac{(1+4\mu^2)^{1/2}(-16\mu^4+24\mu^3-6\mu^2+3\mu+1) + (8\mu^4-12\mu^3+5\mu^3-6\mu+2)}{\mu(4\mu^2-1)(\mu^2-1)} \right] \tag{3.39}$$

Table IV shows the calculate ΔE splitting for a number of states.

Table IV: Tipycal ΔE values for some states, given by expression (3.39).

States	ΔE [eV]		
(n_2, n_1)	$N = 1$	$N = 2$	$N = 3$
(0 , 1)	$2,7 \times 10^{-5}$	$7,4 \times 10^{-5}$	$4,1 \times 10^{-5}$
(1 , 2)	$9,7 \times 10^{-6}$	$3,4 \times 10^{-5}$	$2,2 \times 10^{-5}$
(2 , 3)	$4,5 \times 10^{-6}$	$1,9 \times 10^{-5}$	$1,4 \times 10^{-5}$

Note: These results hold for the case with $\nu = \lambda = 1$.

Finally, let us discuss an interesting feature of our system namely the existence of an electric dipole moment. This is due to the fact that our system violates the discrete symmetries P and T, as discussed by Kazama[14] for the Dirac electron. We shall determine, in our case, the magnitude of the effect by computing the matrix element of the electric dipole moment operator $\vec{d} = e\vec{r}$ for the lowest state with $j = |q| - 1/2$:

$$\langle\vec{d}\rangle_m = e \int d^3x\, \psi_m^+ \vec{r}\, \psi_m / \int d^3x\, \psi_m^+ \psi_m \tag{3.40}$$

where $\psi_m = f(r)\eta_m$, with f given by (3.22) and η_m by (A.10) of the Appendix. The angular part of the integral can be computed, using the Wigner-Eckart theorem and properties P.11 to P.13 of the Appendix. On the other hand, the radial integrals can be calculated making use of the well known integrals involving confluent hypergeometric functions[10]. The final result is

$$\langle d_Z \rangle_m = -\frac{3}{2}\,\frac{q}{|q|}\,\frac{m}{Me\lambda}\,\frac{(n+1)^2}{|q|+\frac{1}{2}} \tag{3.41}$$

where n is the radial quantum number, and the range of m is

$$m = |q|-\frac{1}{2}\ ,\ |q|-\frac{3}{2}\ ,\ldots,\ -|q|+\frac{1}{2} \tag{3.42}$$

It may be remarked that the infinitesimal extra magnetic dipole moment, introduced to resolve the Lipkin, Weisberger Peshkin difficulty, does not alter the result (3.41). Rewritting the result in the usual unities, we have

$$\langle d_Z \rangle_m = -\frac{3}{2}\,\frac{q}{|q|}\,\frac{e}{\lambda\alpha}\,\frac{\hbar}{Mc}\,\frac{m(n+1)}{|q|+\frac{1}{2}} \tag{3.43}$$

where $\hbar/Mc \simeq 3.9 \times 10^{-11}$ cm, is the reduced Compton wave lenght for the electron and a = $e^2 \simeq 1/137$. Using now, the Dirac quantization condition (3.21), then

$$\langle d_Z \rangle_m \simeq -8.01 \times 10^{-9}\,\frac{\nu}{|\nu|}\,\frac{m}{|\nu|+1}\,\frac{(n+1)^2}{\lambda}\ [\mathrm{e.cm}]\ . \tag{3.44}$$

This electric dipole moment, valid for the Pauli electron, is tipically of order 10^{-9} [e.cm].

The analogous result for the spinless electron is

$$\langle d_Z \rangle_m \simeq -5.34 \times 10^{-9}\,\frac{\nu}{|\nu|}\,m\left[(|q|+\frac{1}{4})+\frac{3}{2}\,(|q|+\frac{1}{4})^{1/2}+\frac{1}{2}\right]\ [\mathrm{e.cm}] \tag{3.45}$$

where now the range of m is

$$m = |q| ; |q|-1,\ldots, -|q| \qquad (3.46)$$

This electric dipole moment is also typically of order 10^{-9} $|e.cm|$.

4. CONCLUSIONS

As we have shown[15] in the preceding sections, the monopole harmonics provide a simple and elegant method for treating the Pauli spin equation in the presence of magnetic monopoles.

The results obtained for the two problem discussed in this paper are in agreement with those derived in older literature[4,5]. However, they were here obtained in a much simpler and direct way.

In connection with the electron-dyon system, we have extended previous treatments by calculating fine structure splittings ofthe $j \geq |q| + 1/2$ levels and the expectation value of the electric dipole moment operator for the ground state of the system.

Finally, we wish to stress the desireability of extending the present treatment to the Dirac equation with a repulsive hard-core similar to that empioyed here to estimate the binding energy of a $U(1)$ magnetic monopole to a spin 1/2 atomic nucleus.

APPENDIX

The most relevant definitions and formulas on monopole harmonics are included in this part, specially those on the spinor monopole harmonics that have been used in the text.

The (scalar) monopole harmonics $Y_{q,\ell,m}$ are defined as the simultaneous eigen-sections of the angular momentum operators $\vec{L}^2$ and L_z (see equation 2.3 for the definitions of $\vec{L}$)

$$\begin{aligned} \vec{L}^2 Y_{q\ell m} &= \ell(\ell+1) Y_{q\ell m} \\ L_z Y_{q\ell m} &= m Y_{q\ell m} \end{aligned} \qquad (A.1)$$

where $\ell = 0, 1/2, 1, \ldots,$ and for a given ℓ, $m = -\ell, -\ell+1, \ldots, \ell$. Given q, the possible ℓ values are

$$\ell = |q| \, , \, |q| + 1, \, \ldots \tag{A.2}$$

For explicit expressions of the monopole harmonics, the reader is referred to the papers of Wu and Yang[1].

The spinor monopole harmonics, by the other hand, are defined as simultaneous eigen-sections of the operators $\vec{J}^2$ and J_Z (see equation 2.2 for the definitions of $\vec{J}$)

$$\begin{aligned} \vec{J}^2 \, \Phi^{(i)}_{qjm} &= j(j+1)\, \Phi^{(i)}_{qjm} \\ J_Z \, \Phi^{(i)}_{qjm} &= m \, \Phi^{(i)}_{qjm} \end{aligned} \tag{A.3}$$

where $i = 1,2$ refers to $\ell = j \mp 1/2$ respectively, and with

$$\Phi^{(i)}_{qjm} = \sum_{m_\ell + m_\sigma = m} \left\langle j \pm \tfrac{1}{2} \, m_\ell \, \tfrac{1}{2} \, m_\sigma \middle| jm \right\rangle \, Y_{q,\, j\pm\frac{1}{2},\, m} \, \chi_{m_\sigma} \, . \tag{A.4}$$

Explicitly, we have

$$\Phi^{(1)}_{qjm} = \begin{bmatrix} \left(\dfrac{j+m}{2j}\right)^{1/2} Y_{q,\, j-\frac{1}{2},\, m-\frac{1}{2}} \\ \\ \left(\dfrac{j-m}{2j}\right)^{1/2} Y_{q,\, j-\frac{1}{2},\, m+\frac{1}{2}} \end{bmatrix}$$

and

$$\Phi^{(2)}_{q,j,m} = \begin{bmatrix} -\left(\dfrac{j-m+1}{2j+2}\right)^{1/2} Y_{q,\, j-\frac{1}{2},\, m+\frac{1}{2}} \\ \\ \left(\dfrac{j+m+1}{2j+2}\right)^{1/2} Y_{q,\, j+\frac{1}{2},\, m+\frac{1}{2}} \end{bmatrix} \tag{A.5}$$

where the possible j values are given by

$$j = |q| + N - \frac{1}{2} \tag{A.6}$$

such that, for

$$\Phi^{(1)}_{qjm} \rightarrow N = 1,\ 2,\ 3,\ \ldots \tag{A.7}$$

and for

$$\Phi^{(2)}_{qjm} \rightarrow N = 0,\ 1,\ 2,\ \ldots \tag{A.8}$$

The collection of all $\Phi^{(i)}$'s form a complete orthonormal set of two-components spinor monopole harmonics.

For $q=0$, the monopole harmonics are simply the ordinary spherical harmonics, and the following properties hold:

(P.1) $(\vec{\sigma}.\vec{r})\ \Phi^{(1)}_{qjm} = -r\ \Phi^{(2)}_{qjm}$

(P.2) $(\vec{\sigma}.\vec{r})\ \Phi^{(2)}_{qjm} = -r\ \Phi^{(1)}_{qjm}$

From the expression

$$\vec{L}^2 = \vec{J}^2 - \vec{\sigma}.\vec{L} - \frac{3}{4} \tag{A.9}$$

one has

(P.3) $\vec{L}^2\ \Phi^{(1)}_{qjm} = (j-\frac{1}{2})(j+\frac{1}{2})\ \Phi^{(1)}_{qjm}$; $j = \ell + \frac{1}{2}$, $\ell = 0,1,2,\ldots$

(P.4) $\vec{L}^2\ \Phi^{(2)}_{qjm} = (j+\frac{1}{2})(j+\frac{3}{2})\ \Phi^{(2)}_{qjm}$; $j = \ell - \frac{1}{2}$, $\ell = 1,2,3,\ldots$

When $q\neq 0$, the lowest angular momentum state occurs for $j = |q| - 1/2$ (or $N=0$) and the corresponding angular section is given by

$$\eta_m \equiv \Phi^{(2)}_{qjm}\ ,\ j = |q| - \frac{1}{2}\ . \tag{A.10}$$

In this case, the following properties hold

$$(\text{P.5})\quad (\vec{\sigma}.\vec{r})\eta_m = r\,\frac{q}{|q|}\,\eta_m$$

$$(\text{P.6})\quad \vec{\sigma}.(\vec{p}-Z|e|\vec{A})f(r)\eta_n = -i\,\frac{q}{|q|}\,(\partial_r + r^{-1})f(r)\eta_m\,.$$

For the higher angular momentum states ($j \geq |q|+1/2$ or $N \geq 1$), it is convenient to form[6] the following orthonormal linear combinations of $\Phi^{(1)}_{qjm}$ and $\Phi^{(2)}_{qjm}$

$$\xi^{(1)}_{qjm} = c\Phi^{(1)}_{qjm} - s\Phi^{(2)}_{qjm} \tag{A.11}$$

$$\xi^{(2)}_{qjm} = s\Phi^{(1)}_{qjm} + c\Phi^{(2)}_{qjm} \tag{A.12}$$

where

$$c = \frac{q}{|q|}\,\frac{\left[(2j+1+2q)^{1/2} + (2j+1-2q)^{1/2}\right]}{2(2j+1)^{1/2}} \tag{A.13}$$

and

$$s = \frac{q}{|q|}\,\frac{\left[(2j+1+2q)^{1/2} - (2j+1-2q)^{1/2}\right]}{2(2j+1)^{1/2}} \tag{A. 14}$$

with $c^2 + s^2 = 1$. In this case, the following properties hold :

$$(\text{P.7})\quad (\vec{\sigma}.\vec{r})\,\xi^{(1)}_{qjm} = -r\,\xi^{(2)}_{qjm}$$

$$(\text{P.8})\quad (\vec{\sigma}.\vec{r})\,\xi^{(2)}_{qjm} = -r\,\xi^{(1)}_{qjm}$$

$$(\text{P.9})\quad \vec{\sigma}.(\vec{p}-Z|e|\vec{A})f(r)\quad \xi^{(1)}_{qjm} = i(\partial_r+r^{-1}-\mu r^{-1})f(r)\;\xi^{(2)}_{qjm}$$

$$(\text{P.10})\quad \vec{\sigma}.(\vec{p}-Z|e|\vec{A})g(r)\quad \xi^{(2)}_{qjm} = i(\partial_r+r^{-1}+\mu r^{-1})g(r)\;\xi^{(1)}_{qjm}$$

with $f(r)$ and $g(r)$ arbitrary functions of the distance r, and

$$\mu = \left[N(N + |Z\nu|)\right]^{1/2} \tag{A.15}$$

where we have used the Dirac quantization condition (2.13).

Finally, we give three importants properties[16] of monopole harmonics, that have been used in the text.

$$(P.11)\quad \left(\frac{4\pi}{3}\right)^{1/2} Y_{010} = \cos\theta$$

$$(P.12)\quad Y^*_{q,\ell,m} = (-1)^{q+m}\, Y_{-q,\ell,-m}$$

$$(P.13)\quad \int Y_{q,\ell,m}\, Y_{q',\ell',m'}\, Y_{q'',\ell'',m''}\, d\Omega = \left[\frac{(2\ell+1)(2\ell'+1)(2\ell''+1)}{4\pi}\right]^{1/2} \times$$

$$\times \begin{pmatrix} \ell & \ell' & \ell'' \\ m & m' & m'' \end{pmatrix} \begin{pmatrix} \ell & \ell' & \ell'' \\ q & q' & q'' \end{pmatrix} (-1)^{\ell+\ell'+\ell''}$$

where the large parentheses represent the Wigner $3j$-symbols.

REFERENCES

1. T.T.Wu and C.N.Yang, Phys.Rev. D 12, 3845 (1975) and Nucl. Phys B 107, 365 (1976). For an alternative treatment, see also C.G. Bollini and J.J. Giambiagi, Nucl. Phys. 123 B, 311 (1977) and reference 8 below.
2. P.A.M.Dirac, Proc.Roy.Soc. (London), A *133*, 60 (1931). For an extensive (1973 to 1980) bibliografy on magnetic rnonopoles, see R. A. Carrigan Jr., FERMILAB-77/42 (1973 to 1976) and R.E. Craven, W. P. Trower and R.A. Carrigan Jr., FERMILAB-81/37 (1977 to 1980).
3. C.N.Yang, Ann. N.Y. Aca.Sci. 294, 86 (1977).
4. W.V.R. Malkus, Phys. Rev. 83, 899 (1951).
5. D.Sivers, Phys. Rev. D 2, 2048 (1970).
6. Y.Kazama, C.N.Yang and A.S.Goldhaber, Phys.Rev. D*15*, 2287 (1977).
7. Y.Kazama and C.N.Yang, Phys. Rev. D 15, 2300 (1977).
8. C.G.Bollini and P.L.Ferreira, Nucl.Phys.*137* B, 351 (1978).
9. C.J.Eliezer and S.Roy, Proc. Cambridge Phil. Soc. 58, 401 (1962).
10. L.Landau and E.Lifchitz, *Mécanique Quantique*, Édition Mir, Moscou, 1966. See the appendix f.

11. H.J.Lipkin, W. I.Weisberger and M.Peshkin, Ann.Phys. (NY) 53, 203 (1969)

12. Y.Kazama, C.N.Yang and A.S.Goldhaber, Phys.Rev. D *15*, 2287 (1977).

13. V.Berestetski, E.Lifchitz et L.Pitayevsky, *Théorie Quantique Relativiste*, Première Partie, Editions Mir, Moscou, 1972.

14. Y.Kazama, Phys. Rev. D *16*, 3078 (1977).

15. For an extended version on the present work see J.G.Pereira, "Monopolos magnéticos de Dirac: algumas aplicações do formalismo de Wu e Yang", Tese de Mestrado submetida ao IFT, São Paulo, 1982.

16. T.T.Wu and C.N.Yang, Phys. Rev. D 26, 1018 (1977).