Revista Brasileira de Fisica, Vol. 11, Nº 4, 1981

Glauber Amplitude for Scattering of Charged Particles by Atoms

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Recebido em 20 de Abril de 1981

The $(3Z+1)$-dimensional integral occurring in the Glauber amplitude for scattering of charged particles by Z-electron atoms is reduced to a one-dimensional integral representation involving products of modified Lommel functions. Some explicit results are obtained for the charged particle collision with hidrogen, helium and lithium.

A integral de $(3Z+1)$-dimensões que aparece na amplitude de Glauber para o espalhamento entre as partículas carregadas e os átomos de Z-elétrons é reduzida à integral unidimensional, envolvendo os produtos de funções modificadas de Lommel. Os resultados explícitos são obtidos para o espalhamento entre as partículas carregadas e hidrogênio, hélio e lítio.

1. INTRODUCTION

In a recent paper[1] nonperturbative path-integral approximation, we show that the Glauber amplitude for a charged structureless particle Z'e (with initial velocity v_i and momentum K_i) by an arbitrary Z-electron atom from its initial state ψ_i to its final state ψ_f can be written as

$$F_{fi}(\vec{q}) = \frac{iK_i}{2} \int \psi_f(\vec{r}_1, \ldots, \vec{r}_Z)\; \Gamma(\vec{b};\vec{r}_1, \ldots \vec{r}_Z)\psi_i(\vec{r}_1, \ldots, \vec{r}_Z)$$

$$\times\, e^{i\vec{q}.\vec{b}}\, d^2b\, d\vec{r}_1\, d\vec{r}_2 \ldots\, d\vec{r}_Z \tag{1}$$

Here we have

$$\Gamma(\vec{b};\vec{r}_1,\vec{r}_2,\ldots \vec{r}_Z) = 1 - \prod_{j=1}^{Z} \left(\frac{|\vec{b}-\vec{s}_j|}{b}\right)^{2i\eta} \tag{2}$$

and $\eta = -Z'/v_i$ (in atomic units) by assuming that the interactions of the incident charged particle $(\vec{r})$ with the bound electrons and the nucleus are screened coulombian. In Eqs. (1) and (2), $\vec{q}$ is the momentum transfer vector; $\vec{b}$ and $\vec{s}_j$ are the respective projections of the position vectors of incident charged particles and bound electrons $(\vec{r}_j)$ onto the plane perpendicular to the direction of the Glauber path integration: $\vec{q}$, $\vec{b}$ and $\vec{s}_j$ are all coplanar.

In a previous publication[2] (to be refered to hereafter as TC), new analytical method for reducing the Glauber amplitude[3-5] for charged particle - neutral atom collisions to a one - dimensional integral representation involving modified Lommel functions are proposed. Using these methods, the results for elastic and inelastic charged particle-helium collisions have been reported[2,6-9]. In this paper we wish to point out that these methods can be extended to reduce the the Glauber amplitude (1), a $(3Z+1)$-dimensional integral, to a one-dimensional integral suitabie for numerical integration.

In Sec. 2, we describe the analytical method for reducing the Glauber amplitude (1) to a one-dimensional integral. Then we carry out the calculations for the scattering between charyed particle and hydrogen, helium and lithium in Sec. 3.

2. ANALYTIC METHOD

In this paper, we assume that the product of the bound-state wave functions has the following form[10]

$$\psi_f^*\psi_i = \sum_{k=0}^{N} \left\{ c_k \prod_{j=1}^{Z} \left[r_j^{\mu_{k,j}+\ell_j+\ell'_j} Y_{\ell_j m_j}(\Theta_j,\phi_j) Y_{\ell'_j m'_j}(\Theta_j,\phi_j) \right] \right\} \quad , \tag{3}$$

where (r_j,Θ_j,ϕ_j) are the spherical coordinates of the jth bound-electron and $Y_{\ell m}$ are normalized spherical harmonics[11]. Applying Eqs.(2.5.6), (2.5.17) and *(2.5.29)* of Ref. 11, the Eq. (3) becomes

$$\psi_f^*\psi_i = \sum_{k=0}^{N} \sum_{n_1=0}^{n_{1,\max}} \cdots \sum_{n_Z=0}^{n_{Z,\max}} \left\{ c_k \left[\prod_{j=1}^{Z} \frac{A_j}{\Pi} r_j^{n_{k,j}} e^{-\alpha_{k,j} r_j} s_j^{M_j} e^{iM_j\phi} Z_j^{2\nu_j} \right] \right\}$$

and

$$A_j = (-1)^{m_j+n_j+n'_j} \frac{1}{4} \left[(2\ell_j+1)(2\ell'_j+1) \frac{(\ell_j-m_j)!(\ell'_j+m'_j)!}{(\ell_j+m_j)!(\ell'_j-m'_j)!} \right]^{1/2}$$

$$\times \frac{(2\ell_j - 2n_j)!}{2^{\ell_j}(\ell_j-m_j)!n_j!(\ell_j-m_j-2n_j)!} \cdot \frac{(2\ell'_j - 2n'_j)!}{2^{\ell'_j}(\ell'_j-m'_j)!n'_j!(\ell'_j+m'_j-2n'_j)!} \tag{4}$$

with $n_{j,\max} \leqslant (\ell_j-m_j)/2$ and $n'_{j,\max} \leqslant (\ell'_j+m'_j)/2$. We have, for later convenience, set $M_j=m_j-m'_j$, $n_{k,j}=\mu_{k,j}+2n_j+2n'_j$ and $2\nu_j=\ell_j+\ell'_j-m_j+m'_j-2n_j-2n'_j$.

Substituting Eq. (4) into Eq. (1), we immediately obtain the selection rule

$$F_{fi}(q) = 0 \quad \text{if} \quad (\ell_j+\ell'_j-m_j+m'_j) \quad \text{is odd integer.} \tag{5}$$

We note that the Glauber amplitude can be written as

$$F_{fi}(q) = \frac{iK_i}{2} e^{iM\phi_q} \sum_{k=0}^{N} \sum_{n_1,\ldots,n'_Z} \left\{ c_k \left[\prod_{j=1}^{Z} \left(-\frac{\partial}{\partial\lambda_j}\right)^{n_{k,j}+1} \right] \right.$$

$$\left. \times I(\lambda_1,\ldots,\lambda_Z;q) \quad |\lambda_j = \alpha_{k,j}| \right\} \tag{6}$$

In Eq. (6), we define the generating function via

$$I(\lambda_1,\ldots,\lambda_Z;q) = \frac{1}{\Pi^{Z+1}} \int_0^\infty b\, db \int_0^{2\Pi} d\phi_b\, e^{i\vec{q}\cdot\vec{b}+iM(\phi_b-\phi_q)}$$

$$\times \left\{ \prod_{j=1}^{Z} \left[A_j\, s_j^{M_j+1}\, ds_j \int_0^{2\Pi} e^{iM_j(\phi_j-\phi_b)}\, d\phi_j \right.\right.$$

$$\times \int_0^\infty z_j^{2\nu_j} \frac{e^{-\lambda_j(s_j^2+z_j^2)^{1/2}}}{(s_j^2+z_j^2)^{1/2}}$$

$$\left.\left. \times \left[1 - \prod_{j=1}^{Z} \left(\frac{b^2+s_j^2-2bs_j\cos(\phi_j-\phi_b)}{b^2} \right)^{i\eta} \right] dz_j \right] \right\} \tag{7}$$

where $M = \sum_{j=1}^{Z} M_j$.

Ernploying the standard formulas[12] for J_ν and K_ν,

$$\int_0^{2\Pi} e^{iqb\cos\phi_b + iM\phi_b}\, d\phi_b = 2\, i^M J_M(qb) \,,$$

$$\int_{-\infty}^{\infty} z^{2\nu} \frac{e^{-\lambda(s^2+z^2)^{1/2}}}{(s^2+z^2)^{1/2}}\, dz = 2^{\nu+1} \frac{\Gamma\left(\nu+\frac{1}{2}\right)}{\sqrt{\Pi}} \left(\frac{s}{\lambda}\right)^\nu K_\nu(\lambda s) \,,$$

and then changing variable $s_i \to bs_i$, we find that (7) **can** be written as

$$I(\lambda_1,\ldots,\lambda_Z;q) = 2^{2Z+1} i^M \int_0^\infty b^{2Z+1+M+\sum_{j=1}^{Z}\nu_j} J_M(qb)\, db \left[\prod_{j=1}^{Z} A_j \left(\frac{2}{\lambda_j}\right)^{\nu_j} \right.$$

$$\left. \times \frac{\Gamma(\nu_j+\frac{1}{2})}{\sqrt{\Pi}} \int_0^\infty s_j^{(M_j+\nu_i+1)} K_{\nu_j}(\lambda_j b s_j)\, ds_j \right] \left\{ \delta_{m_1,m_1'} \ldots \delta_{m_Z,m_Z'} \right.$$

$$\left. - \left[\prod_{j=1}^{Z} \frac{1}{2\Pi} \int_0^{2\Pi} e^{iM_j(\phi_j-\phi_b)} (1+s^2-2s_j\cos(\phi_j-\phi_b))^{i\eta}\, d\phi_j \right] \right\} \tag{8}$$

We now utilize the result[12] that

$$\int_0^\infty s^{M+\nu+1} K_\nu(\lambda bs)\ ds = \frac{2^{M+\nu}}{(\lambda b)^{M+\nu+2}} \Gamma\left(\frac{M+2+2\nu}{2}\right) \Gamma\left(\frac{M+2}{2}\right) . \tag{9}$$

Finally, the $(3Z+2)$-dimensional integral (7) has been reduced to the following one-dimensional integral

$$I(\lambda_1,\ldots,\lambda_Z;q) = 2^{2Z+1} i^M \left[\prod_{j=1}^{Z} A_j \left(\frac{2}{\lambda_j}\right)^{\nu_j} \frac{\Gamma(\nu_j+\frac{1}{2})}{\sqrt{\Pi}}\right]$$

$$\times \int_0^\infty b^{2Z+1+M_j+\sum_{j=1}^{Z}\nu_j} J_M(qb)db \left\{\left[\prod_{j=1}^{Z} \frac{1}{4}\left(\frac{2}{\lambda_j b}\right)^{M_j+\nu_j+2} \Gamma\left(\frac{M_j+2+2\nu_j}{2}\right)\right.\right.$$

$$\left.\left.\times\ \Gamma\left(\frac{M_j+2}{2}\right) - \delta_{m_j,m_j'}\right] - \left[\prod_{j=1}^{Z} M_{M_j,\nu_j}(\lambda_j b)\right]\right\} \tag{10}$$

where we define $M_{\mu,\nu}(x)$ via

$$M_{\mu,\nu}(x) = \int_0^\infty s^{\mu+\nu+1} K_\nu(xs)ds\left(\frac{1}{2\Pi}\right) \int_0^{2\Pi} e^{i\mu\phi}\ (1+s^2-2s\ \cos\phi)^{i\eta}\ d\phi \tag{11}$$

We now introduce the integral representation of Thomas and Gerjuoy[14] (Eq.(A6) of Ref. 14, this reference will be refered to hereafter as TG) to replace the integral over ϕ in Eq. (11) by an equivalent integral involving Bessel functions; namely

$$M_{\mu,\nu} = -2^{2i\eta} \frac{\Gamma(1+i\eta)}{\Gamma(1-i\eta)} \int_0^\infty s^{\mu+\nu+1} K_\nu(xs)ds \int_0^\infty dt\ \ t^{-i2\eta} \frac{d}{dt}\left[J_\mu(t)J_\mu(st)\right] \tag{12}$$

The integral over s is simply[15]

$$\int_0^\infty s^{\mu+\nu+1}\ J_\mu(ts)K_\nu(xs)\ ds = 2^{\mu+\nu}\ t^\mu x^\nu\ \frac{\Gamma(\mu+\nu+1)}{(t^2+x^2)^{\mu+\nu+1}} . \tag{13}$$

Hence

$$M_{\mu,\nu} = -2^{2i\eta+\mu+\nu} \frac{\Gamma(1+i\eta)}{\Gamma(1-i\eta)} \Gamma(\mu+\nu+1)\, x^{\nu} \left\{ \frac{1}{2} \left[K_{\mu+1,\mu+\nu,\mu-1}(x) - K_{\nu+1,\mu+\nu,\nu+1}(x) \right] + \mu K_{\mu,\mu+\nu,\mu}(x) - 2(\mu+\nu+1)\, K_{\mu+2,\mu+\nu+1,\mu}(x) \right\} \tag{14}$$

In Eq. (14), we define

$$K_{n,p,m}(x) = \int_0^\infty t^{-2i\eta-1+n} (t^2+x^2)^{-1-p} J_m(t)\, dt \ . \tag{15}$$

It was point out by Thomas and Chan[2] that the integral of class $K_{n,p,m}$ may be written as a finite sum of terms involving modified Lommel functions $L_{\mu,\nu}(ix)$ via TC Eq. (A7)

$$\int_0^\infty t^{-\mu} J_\nu(t)\,(t^2+x^2)^{-1} dt = -(ix)^{-\mu-1} \left[\Gamma\left(\frac{1+\nu-\mu}{2}\right) / \ \Gamma\left(\frac{1+\nu+\mu}{2}\right)\right] 2^{-\mu}\, L_{\mu,\nu}(ix) \tag{16}$$

Substituting Eqs. (11)-(16) into Eq. (10), we then obtain the generating function in terms of integrals over products of one, two, ..., and Z modified Lommel functions. Applying TC Eq. (A7), TG Eq. (B3) and (B5), the integral, which involvesonly one modified Lommel functions may be evaluated in closed form as a finite sum of the hypergeometric functions ${}_2F_1$. This result is understandable if webreak up the full $\Gamma(\vec{b};\vec{r}_1,\ldots,\vec{r}_Z)$ of Eq. (2) in a fashion analogous to the Glauber multiple-scattering expansion[3], namely

$$\Gamma(b;\vec{r}_1,\ldots,\vec{r}_Z) = \sum_{i=1}^{Z} \Gamma_i(\vec{b};\vec{r}_i) - \sum_{i>j} \Gamma_i(\vec{b};\vec{r}_i)\, \Gamma_j(\vec{b};\vec{r}_j) + \ldots + (-1)^{Z-1} \prod_{i=1}^{Z} \left[\Gamma_i(\vec{b};\vec{r}_i)\right] \ , \tag{17}$$

where

$$\Gamma_i(\vec{b};\vec{r}_1) = 1 - (|\vec{b}-\vec{s}_i|)/b)^{i\eta} .$$

If the expansion (17) is used in Eq. (7), we immediately obtain, using the procedure of Thomas and Gerjuoy[14], the terms involving ${}_2F_1$ come from the integral over sum $\sum_{i=1}^{Z} \Gamma_i(\vec{b};\vec{r}_i)$; whereas the integral involving the product over n modified Lommel functions comes from the integral wer the product of n $\Gamma_i(\vec{b};\vec{r}_i)$.

3. RESULTS

To illustrate the method presented in this paper, we consider the following cases:

(1) Scattering by hydrogen atoms ($Z=1$)

A. 1s - ns Transitions

When $\ell=\ell'=m=m'=M=\nu=0$ and A = 1/4, Eq. (10) reduces to

$$I_0(\lambda;q) = 2\int_0^\infty b^3 \; J_0(qb) \left[(\lambda b)^{-2} - M_{0,0}(\lambda b)\right] db . \tag{18}$$

In Eq. (18), $M_{0,0}(\lambda b)$ was derived by Thomas and Chan[2] (TC Eqs. (7a), (10), (11) end (14b), or[16]

$$M_{0,0}(\lambda b) = 2^{i\eta} \frac{\Gamma(1+i\eta)}{\Gamma(1-i\eta)} \left\{ 2\, K_{2,1,0}(\lambda b) + \frac{1}{2}\left[K_{1,0,1}(\lambda b) - K_{1,0,-1}(\lambda b)\right]\right\}$$

$$= (\lambda b)^{-2} + (2i\eta)^2 (i\lambda b)^{-2-2i\eta} \; L_{2i\eta-1,0}(i\lambda b) . \tag{19}$$

Hence

$$I_0(\lambda;q) = \frac{2(2i\eta)^2}{\lambda^2} \int_0^\infty b\; J_0(qb)\,(i\lambda b)^{-2i\eta}\; L_{2i\eta-1,0}(i\lambda b)\,db , \tag{20}$$

which reduces, via TC Eq. (18), to

$$I_0(\lambda;q) = -4i\eta\ \Gamma(1+i\eta)\Gamma(1-i\eta)\lambda^{-2-2i\eta}\ q^{-2+2i\eta}\ {}_2F_1(1-i\eta,1-i\eta;1;-\lambda^2q^{-2}), \quad (21)$$

in agreement with the generating function previously derived by Thomas and Gerjuoy (TG Eq. (19c)). Here Γ and ${}_2F_1$ are the usually gamma and hypergeometric functions, respectively.

B. $1s$ - np Transitions

When $\ell=m=0, \ell'=1, m'=-1$ and $A = \sqrt{3/8}$, Eq. (10) becomes

$$I_1(\lambda;q) = -2\sqrt{6}\ i \int_0^\infty b^4\ J_1(qb)\ M_{1,0}(\lambda b)\ db\ , \quad (22)$$

where $M_{1,0}(\lambda b)$ was derived by Chan and Chen (Eqs. (11) and (18) of ref. 6), namely,

$$M_{1,0}(\lambda,b) = -\ Z^{2i\eta+2}\ \frac{\Gamma(1+i\eta)}{\Gamma(1-i\eta)} \Bigg\{ K_{1,1,1}(\lambda b) - 4\ K_{3,2,1}(\lambda b) - \frac{1}{2}$$

$$\times \left[K_{2,1,1}(\lambda b) - K_{2,1,-1}(\lambda b)\right] \Bigg\}$$

$$= 4i\eta(\lambda b)^{-2}\left[(1+i\eta)(i\lambda b)^{-1-2i\eta}\ L_{2i\eta,1}(ihb) - i\eta(i\lambda b)^{-2i\eta} L_{2i\eta-1,0}(i\lambda b)\right].$$

(23)

Substituting Eq. *(23)* into Eq. (22), and then applying TC Eq. (A7), TG Eq. (B3) and TG Eq. (B5), we obtain

$$I_1(\lambda;q) = 2\sqrt{6} \Bigg\{ 4i\ \Gamma(1+i\eta)\Gamma(2-i\eta)(i\eta)^{-2i\eta-2}\ q^{2i\eta-3}$$

$$\times \left[-\ {}_2F_1(2-i\eta,1-i\eta;1;-\lambda^2q^{-2}) + (1+i\eta)\ {}_2F_1(2-i\eta,1-i\eta;2;-\lambda^2q^{-2})\right]\Bigg\}\ . \quad (24)$$

Eq. (24) differs from TG Eq. (27c) only by a numerical factor, since our $I_1(A;q)$ is defined as $2\sqrt{6}$ $I_1(A;q)$ of TG Eq. (23b).

(2) *Scattering by He atoms* (Z=2)

Setting $l_i=l_i'=m_i=m_i'=\nu_i=0$ (i=1,2) and $A_1=A_2=1/4$, Eq. (10) for 1^1s-n^1s transitions reduces to

$$I_0(\lambda_1,\lambda_2;q) = 2\int_0^\infty b^5 J_0^\bullet(qb)\left[(\lambda_1 b)^{-2}(\lambda_2 b)^{-2} - M_{0,0}(\lambda_1 b)M_{0,0}(\lambda_2 b)\right]db \tag{25}$$

by using Eq. (19) and TC Eq. (18) which leads to

$$I_0(\lambda_1,\lambda_2;q) = -(\lambda_1\lambda_2)^{-2}\,(2i\eta)^2\,\Gamma(i\eta)\Gamma(1-i\eta)q^{2i\eta-2}$$

$$\lambda_1^{-2i\eta}\,{}_2F_1(1-i\eta,1-i\eta;1;-\lambda_1^2 q^{-2}) + \lambda_2^{-2i\eta}\,{}_2F_1(1-i\eta,1-i\eta;1;\lambda_2^2 q^{-2})$$

$$-2(2i\eta)^4(\lambda_1\lambda_2)^{-2}\int_0^m b\,J_0(qb)(i\lambda_1 b)^{-2i\eta}\,L_{2i\eta-1,0}(i\lambda_1 b)(i\lambda_2 b)^{-2i\eta}$$

$$\times L_{2i\eta-1,0}(i\lambda_2 b)\,db = 2^{-4}\,.\quad \text{TC Eq. (19).} \tag{26}$$

(3) *Scattering by* Li *atoms* (Z=3)

Setting $l_3=l_j'=m_j=m_3'=\nu_3=0$ and $A_3=1/4$ (j = 1,2,3), and then using Eq. (19) and Eq. TC (18), we obtain the generating functions for 1^2s-n^2s transitions as[17]

$$I_0(\lambda_1,\lambda_2,\lambda_3;q) = -(\lambda_1\lambda_2\lambda_3)-2\left\{(2i\eta)^2\,\Gamma(i\eta)\Gamma(1-i\eta)q^{2i\eta-2}\times\right.$$

$$\times\sum_{j=1}^{3}\left[\lambda_j^{-2i\eta}\,{}_2F_1(1-i\eta,1-i\eta;1;\frac{\lambda_j^2}{q^2})\right] +$$

$$+\,2(2i\eta)^4\int_0^\infty b\,J_0(qb)\left[(i\lambda_1 b)^{-2i\eta}\,L_{2i\eta-1,0}(i\lambda_1 b)(i\lambda_2 b)^{-2i\eta}L_{2i\eta-1,0}(i\lambda_2 b)\right.$$

$$+ (i\lambda_1 b)^{-2i\eta} L_{2i\eta-1,0}(i\lambda_1 b)(i\lambda_3 b)^{-2i\eta} L_{2i\eta-1,0}(i\lambda_3 b)$$

$$+ (i\lambda_2 b)^{-2i\eta} L_{2i\eta-1,0}(i\lambda_2 b)(i\lambda_3 b)^{-2i\eta} L_{2i\eta-1,0}(i\lambda_3 b)\Big] db$$

$$+ 2(2i\eta)^6 \int_0^\infty b\, J_0(qb) \left[\prod_{j=1}^{3} (i\lambda_j b)^{-2i\eta} L_{2i\eta-1,0}(i\lambda_j b)\right] db \Bigg\} \tag{27}$$

For small-angle scattering, the contributions to the scattering amplitude come mainly from the region where the impact parameter is large and the incident particle does not see much of inner electrons[18]; thus we may use the following approximation

$$\Gamma(\vec{b};\vec{r}_1,\vec{r}_2,\vec{r}_3) \simeq \Gamma_3(\vec{b};\vec{r}_3) \; . \tag{28}$$

It can be easily shown that the generating function can be written as

$$I(\lambda_3;q) = 2^3 i^{M_3} A_3 \left(\frac{2}{\lambda_3}\right)^{\nu_3} \frac{\Gamma(\nu_3+\frac{1}{2})}{\sqrt{\Pi}} \int_0^\infty b^{3+M_3+\nu_3} J_{M_3}(qb)\, db$$

$$\times \left\{ \frac{1}{4} \left(\frac{2}{\lambda_3 b}\right)^{M_3+\nu_3+2} \Gamma\left(\frac{M_3+2+2\nu_3}{2}\right)\Gamma\left(\frac{M_3+2}{2}\right) \delta_{m_3,m_3'} - M_{M_3,\nu_3}(\lambda_3 b) \right\} . \tag{29}$$

For $M_3 = \nu_3 = 0$ case, Eq. (29) becomes Eq. (15) of Ref. 17 by using Eqs. (19), (20) and TG (18).

Finally, we present an even simpler result for the interaction of the charged particle, with the atomic electrons and nucleus having the following form[19,20] (valid for alkali atoms)

$$V(\vec{r};\vec{r}_1,\ldots,\vec{r}_Z) = -\frac{1}{r} + \frac{1}{|\vec{r}-\vec{r}_Z|} - \frac{A\exp(-\lambda r)}{r} \quad , \tag{30}$$

where $\vec{r}_Z$ is the coordinate of the valence electron. In Eq. (30), A and A are respectively, the depth and the range of the effective potential between the charged particle and $(Z-1)$ inner electrons and $(Z-1)$ protons of atom. Then we have

$$\Gamma(\vec{b};\vec{r}_1,\ldots,\vec{r}_Z) = \Gamma(\vec{b};\vec{r}_Z) = 1 - (|\vec{b}-\vec{s}_Z|/b)^{2i\eta} \quad \exp \frac{2iAK_0(\lambda b)}{K_i} \tag{31}$$

Substituting (31) into (7), we can easily obtain the following generating function

$$I(\lambda_Z;q) = 2^3 \, i^{M_Z} A_Z \left(\frac{2}{\lambda_Z}\right)^{\nu_Z} \frac{\Gamma(\nu_Z+\frac{1}{2})}{\sqrt{\Pi}} \int_0^\infty b^{3+M_Z+\nu_Z} \, J_{M_Z}(qb) \, db$$

$$\times \left\{ \frac{1}{4} \left(\frac{2}{\lambda_Z b}\right)^{M_Z+\nu_Z+2} \Gamma\left(\frac{M_Z+\nu_Z+2}{2}\right)\Gamma\left(\frac{M_Z+2}{2}\right)\delta_{m_Z,m_Z'} - M_{M_Z,\nu_Z}(\lambda_Z b)\right.$$

$$\left. \times \exp\left[\frac{2iAK_0(\lambda b)}{K_i}\right]\right\} . \tag{32}$$

Now we consider the charged particle - lithium scattering with $M_Z=\nu_Z=0$, then we have

$$I_0(\lambda_3;q) = 2 \lambda_3^{-2} \int_0^\infty b \, J_0(qb) \left\{ 1 - \left[1 - (2i\eta)^2 (i\lambda_3 b)^{-2i\eta} L_{2i\eta-1,0}(i\lambda_3 b)\right.\right.$$

$$\left.\left. \left[\frac{2iAK_0(\lambda b)}{K_i}\right]\right]\right\} db \quad , \tag{33}$$

after lengthy but straighforward calculations. Using $K_0(\lambda b) \rightarrow 0$ as $b \rightarrow \propto$, we obtain

$$I_0(\lambda_3;q) = 2(2i\eta)^2 \lambda_3^{-2} \int_0^\infty b \, J_0(qb)$$

$$\times \exp\left[\frac{2iAK_0(\lambda b)}{K_i}\right] (i\lambda_3 b)^{-2i\eta} L_{2i\eta-1,0}(i\lambda_3 b) \, db \quad , \tag{34}$$

since the main contributions come from the region where the impact parameter is large. When $A \rightarrow 0$, Eq. (34) will be reduced to Eq. (20) as we expect.

The work of F.T.Chan was supported in part by the Arkansas EPSCOR grant from the U.S. National Science Foundation.

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