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An Inversion to O'Raifeartaigh's Theorem

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The following question is asked: What should be the structure of a space-time symmetry S if, in a finite Lie group combination with an internal symmetry J, the multiplets of particles defined by the unitary representations of S have the same mass? The answer is given by a set of conditions on the Lie algebra of S. When these conditions are avoided it is possible to derive a class of space-time symmetries which admit non zero multiplet mass splitting. The particular cases of semi simple groups $SO(r,s)$ are studied.

Pergunta-se: qual seria a estrutura de uma simetria do espaço-tempo S se, em um grupo de Lie finito combinado com uma simetria interna J, os multipletos das partículas definidas pelas representações unitárias de S tivessem a mesma massa? A resposta é dada por um conjunto de condições sobre a álgebra de Lie de S. Quando essas condições são evitadas é possível deduzir uma classe de simetrias do espaço-tempo que admitem uma separação não nula das massas de um multipleto. Os casos particulares de grupos semisimples $SO(r,s)$ são estudados.

The algebraic properties of the Poincaré group together with the analytic properties of the mass operator have produced several theorems showing that the unification of the Poincaré and internal groups of symmetries cannot agree with the experimental evidence[1,2]. These results are generalized by a well known theorem by L. O'Raifeartaigh[3] and its more precise reformulations[4,5]. It states that a finite dimensional Lie group P generated by a combination of the Poincaré group and an internal symmetry group J gives always a zero squared mass difference for particles belonging to the same spin multiplet.

There are three basic ways to avoid the mentioned theorem: (a) to assume that G is an infinite Lie group, (b) admitting that G is not a Lie group and finally, (c) replacing P by another suitable space--time symmetry. The alternative (a) has been explored to a certain extent but it faces the difficulties inherent to the classification ofthe representations of infinite Lie groups[6]. In this case an extended version of the theorem have been proposed[7]. The alternative (b) finds its most expressive example in the current supersymnetric theories whers G is assumed to be a graded Lie algebra[e].

In this note the potentialities of the alternative (c) are investigated. More precisely the arguments of O'Raifeartaigh's theorem are inverted so that the structure of a space-time symmetry S may be obtained under the supposition that no mass splitting will occur when S and J are cornbined in a finite Lie group G. Then by exclusion, it would be possible to find the type of symmetry which is compatible with a non zero mass splitting.

In the case of P the mass operator is defined by the secund order Casimir operatur

$$\bar{M}^2 = \eta_{ab}\, p^a\, p^b \qquad a\,,\, b = 1,\ldots,4 \tag{1}$$

where η_{ab} is the Minkowski rnetric and p^a the momentum operators. Expression (1) suggests that the mass operator M^2 for S should be taken to a quadratic form defined in the universal envelloping algebra $E(S)$ of S, composed with a given collection of generators of the Lie algebra of S such that: (1) M^2 commutes with all elements of S (that is $\bar{M}^2$ belongs to the center of $E(S)$); (2) M^2 is self adjoint when regarded as an operator acting on a Hilbert space and (3) when S reduces to P by a given rule then M^2 should reduce to (1) following the same rule. With these conditions M^2 may qualify as a niass operator with a continuous spectrum containing real isolated points.

It is not the purpose of this note to discuss the functional analysis aspects of the mass operator as this has been the subject of substancial discussion[9]. Here the main interest lies on the algebraic aspects of the problem.

The paper has been organized as follows: Theorem 1 states the algebraic conditions which S has to satisfy to produce a zero mass splitting in a specified form a combination with J. Theorem 2 shows which conditions S has to satisfy when no mass splittinq arises with an arbitrary combination. The followinq corolary states that a semi simple group cannot satisfy theorem 2 unless it is Abelian. At the end the paper presents an application of the last corolary, where an expression for the mass splitting resulting from S = $SO(r,s)$, is obtained.

It is assumed that the generators of S may be rearranged so that they are separated in two classes: the generators M_a entering in the composition of M^2 and the remaining ones denoted by L_i (boldface Latin indices run from h + 1 to the dimension of the algebra). Therefore, according to the mentioned properties the mass operator may be written as

$$M^2 = a^{ab} M_a M_b \tag{2}$$

where $a^{ab} = a^{ba}$ are certain coeficients not all zero. Notice that the M_a's are not required to commute.

The generators of the combined symmetry G not belonging to S are denoted by N_A (capital Latin indices assume values which are different from these in the indices of the generators of S). Therefore the generators of G are M_a, L_i and N_A. When unspecified, an arbitrary generator of G will be denoted by E_α (Greek indices running through the entire Lie algebra of G). Therefore E_α may be either M_a, L_i or N_A.

Summation convention is applied throughout to all kinds of indices with their respective range. For simplicity the same letters are used to designate the groups and the respective Lie algebras. The following general lemma results from spectral theory in Hilbert spaces[10].

Lemma. Consider a self adjoint operator M^2 acting on a Hilbert space R. Cet E be another operator acting on H such that at least one eigenstate $|b\rangle$ of M^2 lies in the domain of E. Then:

$$\langle a|V_k(E)|b\rangle = (m_a^2 - m_b^2) \quad \langle a|E|b\rangle \ , \tag{3}$$

where m_a^2 is the eigenvalue of M^2 corresponding to the eigenstate $|c>$ and

$$V_k(E) = [M^2, V_{k-1}(E)] \ , \ V(E) = E, \tag{4}$$

$k>a$ integer

The expression (4) for $V_k(E)$ can be written in terms of the adjoint representation of G. This follows from a simple induction on k: If $D(E_a)$ denotes the elements of the adjoint representation of G, then $D(E_\alpha) = [E_\alpha, E]$ and

$$V_1(E) = [M^2, E] = a^{ab} M_a D(M_b)E = a^{ab} D(M_b)D(M_a)E + a^{ab}D(M_a)E \ ,$$

or

$$V_1(E) = b(1)^{ab} M_a D(M_b)E + b(2)^{ab} D(M_b)E \ ,$$

where $b(1)^{ab} = 2\, b(2)^{ab} = a^{ab}$.

Now

$$V_k(E) = [M^2, V_{k-1}(E)] = a^{mn} M_m[M_n, V_{k-1}(E)] + a^{mn} [M_m, V_{k-1}(E)] M_n$$

Taking $k=2$ and applying the above expression for $V_1(E)$, and expression for $V_2(E)$ in terms of $D(M_a)$ is obtained. The result is then replaced in the expression for $V_3(E)$ and so forth. Then it follows that for any k,

$$V_k(E) = \sum_{\rho=k}^{2k} b(e)^{ab\cdots c|de\cdots f} M_a M_b \cdots M_c D(M_d)D(M_e)\cdots D(M_f)E \ , \tag{5}$$

where the coefficients $b(e)^{ab\cdots c|de\cdots f}$ are cornbinations of products of a^{ab}. They have a total of $2k$ indices with $p-k$ indices $a, b \ldots c$ and $3k-\rho$ indices $d, e \ldots f$.

Theorem 1. Let G a Lie algebra of finite dimension or of infinite but numerable dimension. Let S be a subalgebra of G such that its universal enveloping a'gebra contains the operator (2). For a given integer N, the sufficient condition for which $V_k(E) = 0$, $k>N$, is that (N pairs of brackets):

$$[M_d\,[M_e\,\ldots\ldots[M_f, E_\alpha]]\ldots] = 0\ . \tag{6}$$

It follows from (5) that $V_k(E) = 0$, $k > N$, is satisfied for $N > 1$, whenever the following condition is imposed (N factors):

$$D(M_d)\,D(M_e)\,\ldots\ldots\,D(M_f)E = 0 \tag{7}$$

Since $V_k(E)$ is a linear combination of $V_k(E_\alpha)$ then $V_k(E_\alpha) = 0$, $k > N$ implies $V_k(E) = 0$, $k > N$. Therefore the condition (6) is sufficient to $V_k(E) = 0$, $k > N$.

Notice that S is a subalgebra of G so that the following conditions on the structure constants of G must also be considered:

$$C^A_{\underset{\sim}{ij}} = 0\ ,\quad C^A_{\underset{\sim}{i}a} = 0\ ,\quad C^A_{ab} = 0\ . \tag{8}$$

Replacing E_α by M_a, L_i, N_a in (6) gives respectively

$$\begin{aligned} &[M_d\,[M_e\,\ldots\,[M_f, M_a]\,\ldots]] = 0\\ &[M_d\,[M_e\,\ldots\,[M_f, L_i]\,\ldots]] = 0\\ &[M_a\,[M_e\,\ldots\,[M_f, N_a]\,\ldots]] = 0 \end{aligned} \tag{9}$$

The first two equations refer to the structure of S while the last equation refers to the immersion of S in G.

It follows, from (3) that the equations (9) represent, so to speak the strongest set of conditions required from a geometrical symmetry S, so that no mass splitting occurs in a exact combined symmetry scheme where a finite N is especified. For example taking N=2 the conditions (9) become

$$[M_e[M_f, M_a]] = 0,\quad [M_e[M_f, L_i]] = 0,\quad [M_e[M_f, N_a]] = 0, \tag{10}$$

Unlike the trivial case N=1, two types of Lie algebras satisfying (10) can be found. In the first solution the generators M_a belong to an Abelian normal subgroup of S and it may contain the Poincaré group in a particular case. The second solution does not contain a subgroup of S generated by the M_a and for that reason it could be regarded as unphysical. The last equation (9) says that the integer N is anindicatorof the form of immersion of S in G. As N increases the equations (9) become more difficult to be solved but the two rnentioned types of solutions always emerge.

For the more interesting case of an unspecified imersion of S on G the value of N should be taken to be arbitrariiy large. In this case it is interesting to ask if it is possible to find a symmetry S which does not satisfy (9). For an arbitrary Lie algebra G this may be difficul: to answer but if G is an algebra of finite dimension an answer may be given. Such limitation is a consequence of the nilpotency of a linear combination of the operators $D(M_a)$. The following statements deal with such situation.

<u>Lemma</u>: Let G be a finite Lie algebra containing anon Abelian sub algebra S. The necessary and sufficient conditions for the existence of constants b^a such that $b^a \; D(M_a)$ is nilpotent are

$$[D(E_\alpha),\; D(E\beta)] = C^a_{\alpha\beta} D(M_a) \;, \quad C^a_{\gamma\delta}[D(M_a), D(E_\alpha)] = 0 \quad (11)$$

where E_α, E_β are operators of S, chosen among M_α and L_i and $C^\gamma_{\alpha\beta}$ denote the structure constants of G.

Assuming the existente of the constants b^a, the nilpotency condition requires an integer N ruch that $(b^a \; D(M_a))^n = 0$, $n>N$. Therefore

$$\text{tr} \; (b^a \; D(M_a))^n = 0 \qquad n = 1,2 \; \ldots \qquad (12)$$

Now under the hypothesis tnat G is finite, the operators $D(M_a)$ are also finite and therefore (12) says that for any value of n $(b^a \; D(M_a))^n$ is equivalent to a linear combination of commutators.

From the Lie algebra of S:

$$[E_\alpha, E_\beta] = C^{i}_{\alpha\beta} L_i + C^{a}_{\alpha\beta} M_a \; , \tag{13}$$

where E_α, E_β are either L_i or M_b. ret $\tau^{\alpha\beta}$ be the elements of an invertible matrix defined by $b^a = \tau^{\alpha\beta} C^{a}_{\alpha\beta}$. Then from (13):

$$T^{\alpha\beta}[D(E\alpha), D(E\beta)] = T^{\alpha\beta} C^{i}_{\alpha\beta} D(L_i) + b^a D(M_a) \; .$$

Since $b^a D(M_a)$ is a linear combination of commutators it follows that $C^{i}_{\alpha\beta} D(L_i) = 0$. Therefare

$$[D(E_\alpha), D(E_\beta)] = C^{a}_{\alpha\beta} D(Ma) \; . \tag{14}$$

Furthermore, from this expression

$$(b^a D(M_a))^n = \tau^{\alpha\beta}[D(E_\alpha), D(E_\beta) \;\; (b^a D(M_a))^{n-1}]$$

$$+ \tau^{\alpha\beta} D(E_\beta) \;\; [(b^a D(M_a))^{n-1}, D(E_\alpha)] \; .$$

Again the right hand side of this equation mcst be a linear combination of commutators only. Therefcre set

$$D(E_\beta)[(b^a D(M_a))^{n-1}, D(E_\alpha)] = 0 \; .$$

Taking n=2 the non Abelian solution of the above equation requires that $b^a D([M_a, E_\alpha]) = 0$, or equivalently,

$$C^{a}_{\gamma\delta} \;\; [D(M_a), D(E_\alpha)] = 0$$

Reciprocally, suppose that the conditions (11) hold trus and define the constants $b^a = \tau^{\alpha\beta} C^{a}_{\alpha\beta}$ where again $\tau^{\alpha\beta}$ is an invertible matrix. Thcn from (14)

$$b^a D(M_a) = \tau^{\alpha\beta} \;\; [D(E_\alpha), D(E_\beta)]$$

Taking the n^{th} power of this eqcation and considering (11) it follows that

$$(b^a D(M_a))^n = \tau^{\alpha\beta}[(D(E_\alpha), D(E_\beta)(b^a D(M_a))^{n-1}]$$

Therefore $\mathrm{Tr}(b^a D(M_a))^n = 0$ which implies

$$(b^a D(M_a))^n = 0 \quad , \quad n > N \tag{15}$$

Theorem 2: If S is arbitrarily contained in a finite dimensional Lie group G and satisfying (11) then no mass splitting results.

In fact, the nilpotency condition (15) can be written as ($2n > 2N$ factors):

$$b^a b^B \ldots . b^e \ D(M_a)\ D(M_b)\ \ldots .\ D(M_e) = 0. \tag{16}$$

Since the constants b^a depend on an arbitrary matrix $\tau^{\alpha\beta}$, they can be chosen so that the left hand side of (16), multiplied by E equals expression (5). Therefore, $V_n(E) = 0$, $n > N$ and from (3) it follows that the squared mass difference vanishes.

The Poincare algebra P is one example of a Lie algebra satisfying (11) with $E_\alpha = M_a = P_a$, where P_a are the momentum operators and $E_\beta = L_i$ the generators of the Lorentz subalgebra. The mass operator in this example is given by (1). Consequently no mass splittingcan be obtained with arbitrary finite Lie group combinations of P with internal symmetries. This is the content of O'Raifeartaigh's theorem.

Corolary. If S is semi-simple the only possible solution of (11) is Abelian.

Since there are no Abelian invariant proper subgroups in S, all generators of S can enter in the composition of M^2 which may be written as

$$M^2 = g^{ab} M_a M_b ,$$

where

$$g_{bc} = C^d_{ab} C^a_{dc} , \quad g^{ab} g_{bc} = \delta^a_c$$

and C^{a}_{bc} are the structure constants of S. Since in this case all generators of S are of the type M_a, the equations (11) reduce to

$$[D(M_a), D(M_b)] = C^{c}_{ab} D(M_a) , \quad C^{a}_{bc}[D(M_a), D(M_d)] = 0 \tag{17}$$

The first of these equations expresses only the Lie algebra of S in its adjoint representation. The second equation implies that S is Abelian. This result shows'that it is possible to obtain a multiplet mass splitting with a geometrical symmetry which is semi-simple and non Abelian.

From (6) and (3) it follows that to obtain a non zero mass splitting S should not be a normal subgroup of G. This condition will become clear in the following example.

Consider the semi-simple non Abelian group S = $SO(r,s)$ acting on a pseudo Euclidean space with metric signature $r+s$. Let $\eta_{\mu\nu}$ denote the Cartesien components of the metric tensor in the mentioned space (here Greek indices run from 1 to p, the dimension of the space). If $L_{\mu\nu}$ denote the generators of $SO(r,s)$, the proposed mass operator is

$$M^2 = \eta^{\rho\mu} \eta^{\nu\sigma} L_{\mu\nu} L_{\rho\sigma} \quad . \tag{18}$$

It is interesting to notice that in the case of the de Sitter group $SO(4,1)$ which is contractible in the Poincaré group, the above mass operator reduces to (1) along with the contraction procedure .

Let $\{\lambda\} = \{\lambda_1, \lambda_2 \ldots \lambda_\nu\}$ be a set of eigenvalues of the ν Casimir operators of $SO(r,s)$. Denote by $H_{\{\lambda\}}$ the Hilbert representation space of an unitary representation of $SO(r,s)$. Taking the direct sum of all possible such spaces, a larger Hilbert space H is obtained. The combined symmetry G is taken to be a finite Lie group containing $SO(r,s)$ as a subgroup and such that H is a representation space for G. Thus this representation of G is completely reducible respect to $SO(r,s)$. M^2 is assumed to be self adjoint as an operator in H. A multiplet of $SO(r,s)$ in G is described by each base state vector in $H_{\{\lambda\}}$. Objects belonging to distinct multiplets have a squared mass difference given by (3) with k = 1 and $\langle a|E|b\rangle \neq 0$:

$$m_a^2 - m_b^2 = \frac{\langle a|[M^2,E]|b\rangle}{\langle a|E|b\rangle} \tag{19}$$

whore $|a\rangle$, $|b\rangle$ are state vectors helonginq to distinct representations of $SO(r,s)$. Denote an arbitrary generator of G by E_{ab} which can be in particular a generator of S,$L_{\mu\nu}$; a generator not in S, N_{AB} such that $[L_{\mu\nu},N_{AB}] = 0$ and finally, since S is not a normal subgroup cf G, E_{ab} can also be a nixing generators $M_{\mu A}$ such that

$$[L_{\rho\sigma},M_{\mu A}] = C^{\varepsilon}_{\rho\sigma\mu A}\, L_{\tau\varepsilon} + C^{\tau C}_{\rho\sigma\mu A}\, M_{\tau C}$$

If E is an operator of C not in $SO(r,s)$ it is a linear combination of $M_{\rho A}$ and N_{AB} only. Therefore

$$[L_{\mu\nu},E] = a^{\rho A}\, C^{cd}_{\mu\nu\rho A}\, E_{cd} + b^{AB}_{\mu\nu}\, C^{cd}_{\mu\nu AB}\, E_{cd}$$

Consequently

$$[M^2,E] = \eta^{\mu\rho}\,\eta^{\nu\sigma}\,(a^{\tau A}\, C^{cd}_{\rho\tau A} + b^{AB}\, C^{cd}_{\rho\sigma AB})\{L_{\mu\nu},E_{cd}\}$$

where { , } denotes the anticommutator. Replacing in (19) and denoting

$$\langle a|L_{\mu\nu}|b\rangle = \delta^b_a\, S^b_{\mu\nu} \quad \text{(no sum on } b\text{)} ,$$

the squared mass difference expression becornes:

$$m_a^2 - m_b^2 = \eta^{\mu\rho}\,\eta^{\nu\sigma}\,(S^a_{\mu\nu} + S^b_{\mu\nu}) \Bigg[(a^{\tau A} C^{\varepsilon C}_{\rho\sigma\tau A} + b^{AB}\, C^{C}_{AB}) \frac{\langle a|M_{\varepsilon C}|b\rangle}{\langle a|E|b\rangle}$$

$$+ (a^{\tau A} C^{CD}_{\rho\sigma\tau A} + b^{AB} C^{CD}_{\rho\sigma AB}) \frac{\langle a|N_{CD}|b\rangle}{\langle a|E|b\rangle} \Bigg] \tag{20}$$

It is clear that this expression vanishes whenever $SO(r,s)$ is a normal subgroup of G. Note the nature of the two contributing terms one of them depends only on the mixing generators N_{AE} while the other depends on the generators $L_{\mu\rho}$ cf the internal group.

In thiç example no explanation about the nature of the space -time was given. The case where tne space of the group $SO(r,s)$ is associated with a the embedding space of the space-time may be of interest. The particular case of $SO(4,1)$ corresponds to a class of space-times with constant curvature of the de Sitter type. The maximal numberofparaneters of a space-time isometry occurs only on this situation. Furthermore in the flst limit of such space-time the Poincaré group may be recovered by group contraction and the operator (18) reduces to the usual mass operator[11].

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