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The Green's Function for the N-Dimensional Isotropic Harmonic Oscillator

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Closed representatíons for the Green's functions of the Schrödinger differential equation with a N-dimensional isotropic harmonic oscillator potential are obtained by means of the Sturm-Liouville expansion method.

Calcula-se representações fechadas para a função de Green para a equação diferencial de Schrödinger com um potencial tipo oscilador harmônico isotrópico N-dimensional pelo método de expansão tipo Sturm-Liouville.

1. INTROIDUCTION

The present paper contains a derivation of the Green's function for the Schrödinger differential equation for the N-dimcnsional isotropic harmonic oscillator. Titchmarsh[1] calculated the Green's function for the Schrödinger differential equation with one-dimensional isotropic harmonic oscillator by means of the Sturm-Liouville expansion. A closed representation for the tri-dimensional case was calculated bymeans of the Sturm-Liouville expansion in terms of Whittaker's function[2]. An integral representation for N-dimensional isotropic harmonic oscillator was calculated by Bellandi Filho and Caetano Neto[3] using a spectral decomposition of the Green's function in terms of the harmonic oscillator wave function. We

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calculate for the Schrödinger differential equation with one N-dimensional isotropic harmonic oscillator potential the Green's function by means of the Sturm-Liouville expansion in section 2 where we present a closed integral representation and in section 3 is showed how to obtain a closed representation in terms of the product of two Whittaker's functions.

2. INTEGRAL REPRESENTATION

The Sturm-Liouville method[4] consists in writing the Green's function in terms of two linearly independent solutions of the corresponding homogeneous equation.

The Green's function for the Schrödinger differential equation with one N-dimensional isotropic harmonic oscillator potential satisfies the following inhomogeneous differential equation

$$\left(-\frac{\hbar^2}{2\mu}\nabla_N^2 + \frac{kr^2}{2} - E\right) G^N(\vec{r},\vec{r}\,';E) = -\delta(\vec{r}-\vec{r}\,') \tag{1}$$

where ∇_N^2 is the N-dimensional Laplacian operator.

To apply the Sturm-Liouville method we shall firstly make one partial wave expansion for the Green's function

$$G^N(\vec{r},\vec{r}\,';E) = \sum_{\ell=0}^{\infty} \pi^{-N/2}\,\frac{\Gamma(N/2)}{N-2}\,(\ell-1+N/2)\; C_\ell^{(N-2)/2}(\cos\theta)\,G_\ell(r,r';E) \tag{2}$$

where $C_\mu^\nu(\gamma)$ are the Gegenbauer polynomials and $G_\ell(r,r';E)$ is the radial Green's function that satisfies the following differential equation

$$\left\{-\frac{\hbar^2}{2\mu}\left(\frac{d^2}{dr^2} + \frac{N-1}{r}\frac{d}{dr} - \frac{\ell(\ell+N-2)}{r^2}\right) + \frac{kr^2}{2} - E\right\} G_\ell(r,r';E) =$$

$$= -(rr')^{(1-N)/2}\,\delta(r-r')\ . \tag{3}$$

The solution of this differential equation calculated by means of the Sturm-Liouville expansion is

$$G_\ell(r,r';E) = \frac{1}{\hbar\omega}\,(rr')^{-N/2}\,\frac{\Gamma(\ell/2+N/4-\ /2)}{\Gamma(\ell+N/2)}\times$$

$$\times M_{\frac{\lambda}{2}\,;\,\frac{1}{2}(\ell+N/2-1)}\left(\frac{\mu\omega r^2}{\hbar}\right) W_{\frac{\lambda}{2}\,;\,\frac{1}{2}(\ell+N/2-1)}\left(\frac{\mu\omega r'^2}{\hbar}\right) \tag{4}$$

where $\lambda = E/\hbar\omega$; $M_{\mu,\nu}(x)$ and $W_{\mu,\nu}(x)$ are the Whittaker's functions, M is regular as $r\to 0$ and W is regular as $r'\to\infty$, and $r' > r$.

To calculate the total Green's function we write the product of the Whittaker's functions in one integral representation[5] and introduce in eq.(2) and we obtain

$$G^N(\vec{r},\vec{r}\,';E) =$$

$$= \frac{2\mu}{h^2}\,(rr')^{-\frac{1}{2}(N-2)}\,\frac{\Gamma(N/2)}{(N-2)}\,\pi^{-N/2}\int_0^1 d\xi\;\xi^{-\lambda}\,(1-\xi^2)^{-1}\,e^{-\frac{\mu\omega}{2\hbar}(\vec{r}^{\,2}+\vec{r}\,'^2)\frac{1+\xi^2}{1-\xi^2}}\times$$

$$\times\sum_{\ell=0}^{\infty}(\ell+N/2-1)\;C_\ell^{(N-2)/2}(\cos\theta)\;I_{\ell+N/2-1}\left(2rr'\,\frac{\mu\omega}{h}\,\frac{\xi}{1-\xi^2}\right). \tag{5}$$

To performe the sum we can use the following Neumann expansion[6]

$$\sum_{\ell=0}^{\infty}(\ell+N/2-1)\;C_\ell^{(N-2)/2}(\cos\theta)\;I_{\ell+N/2-1}(z) = \left(\frac{z}{2}\right)^{\frac{1}{2}(N-2)}\frac{\exp(z\cos\theta)}{\Gamma(N/2-1)} \tag{6}$$

where $I_\nu(x)$ is the modified Bessel function and therefore

$$G^N(\vec{r},\vec{r}\,';E) =$$

$$= \frac{1}{\hbar\omega}\left(\frac{\mu\omega}{\pi\hbar}\right)^{N/2}\exp\left\{-\frac{\mu\omega}{2\hbar}(\vec{r}^{\,2}+\vec{r}\,'^2)\right\}\cdot$$

$$\cdot\int_0^1 d\xi\quad\xi^{-\lambda+N/2-1}\,(1-\xi^2)^{-N/2}\,\exp\left\{\frac{\mu\omega}{\hbar}\left(\frac{2\vec{r}\cdot\vec{r}\,'}{1-\xi^2}\xi-\frac{\vec{r}^{\,2}+\vec{r}\,'^2}{1-\xi^2}\,\xi^2\right)\right\} \tag{7}$$

with $\mathrm{Re}(-\lambda+\ /2-1) > 0$.

This expression is an integral representation for the total Green's function for the Schrödinger differential equation which the N-dimensional isotropic harmonic oscillator and it is the same as obtained by Bellandi Filho-Caetano Neto[4] using a generalized Mehler formula.

3. CLOSED REPRESENTATION

In order to get a closed representation for the total Green's function $G^N(\vec{r},\vec{r}\,';E)$ in terms of functions, we change the variable $\operatorname{csh} v = \frac{1+\xi^2}{1-\xi^2}$ in eq. (5)

$$G^N(\vec{r},\vec{r}\,';E) =$$

$$= \frac{\mu}{\hbar^2}\,(rr')^{-N/2+1}\,\frac{\Gamma(N/2)}{N-2}\,\pi^{-N/2}\int_0^\infty dv\,\operatorname{cth}^\lambda v/2\,\exp\left\{-\frac{\mu\omega}{2\hbar}(\vec{r}^{\,2}+\vec{r}\,'^2)\operatorname{csh} v\right\}$$

$$\cdot\sum_{\ell=0}^\infty(\ell+N/2-1)\,C_\ell^{(N-2)/2}(\cos\theta)\,I_{\ell+N/2-1}\left(\frac{\mu\omega}{\hbar}rr'\operatorname{sh} v\right) \tag{8}$$

and we note that the second side of the eq. (6) can be written in the following way

$$\frac{\Gamma(1/2)}{\Gamma(N/2-1)}\left(\frac{xy}{2}\right)^{\frac{1}{2}(N-1)}x^{-\frac{1}{2}(N-2)}\{I_{1/2}(x\ )+I_{-1/2}(xy)\} \tag{9}$$

where $x=\cos\theta$ and $y=\frac{\mu\omega}{\hbar}rr'\operatorname{sh} v$.

If the number N is odd we çan obtain a closed representation for the total Green's function by using the following relation for the modified Bessel function[6]

$$\left(\frac{d}{zdz}\right)^m\{z^{-\nu}I_\nu(z)\} = z^{-\nu-m}\,I_{\nu+m}(z) \tag{10}$$

where $m=\frac{1}{2}(N-1)$ is integer and non-negative number, and therefore

$$G^N(\vec{r},\vec{r}\,';E) = \frac{\mu}{2\hbar^2}(2\pi)^{-\frac{1}{2}(N-1)} x\left(\frac{d}{xdx}\right)^{\frac{1}{2}(N-1)} x^{\frac{1}{2}(N-2)}$$

$$\cdot \int_0^\infty dv \, \mathrm{cth}^\lambda v/2 \; \exp\{-\frac{\mu\omega}{\hbar}(\vec{r}^{\,2}+\vec{r}\,'^2)\mathrm{csh}v\} \; \{I_{\frac{1}{2}(N-2)}(xy) +$$

$$+ I_{-\frac{1}{2}(N-2)}(xy)\} . \qquad (11)$$

The integral in eq. (11) reproduces a product of two Whittaker's functions" and the final expression for the total Green's function is

$$G^N(\vec{r},\vec{r}\,';E) = \frac{1}{2\hbar\omega}(2\pi)^{-\frac{1}{2}(N-1)} \Gamma(-\lambda/2+N/4)\Gamma(-\lambda/2-N/4+1)$$

$$\cdot (\xi\xi')^{\frac{1}{2}(3-N)} \left\{\frac{d}{d(\xi\xi')}\right\}^{\frac{1}{2}(N-1)} (\xi\xi')^{\frac{1}{2}(N-4)}$$

$$\cdot N_{\frac{\lambda}{2};\frac{1}{4}(N-2)}\left(\frac{\mu\omega}{\hbar}\xi^2\right) W_{\frac{\lambda}{2};\frac{1}{4}(N-2)}\left(\frac{\mu\omega}{\hbar}\xi'^2\right) \qquad (12)$$

where we have defined

$$N_{\frac{\lambda}{2};\frac{1}{4}(N-2)}\left(\frac{\mu\omega}{\hbar}\xi^2\right) = \frac{M_{\frac{\lambda}{2};\frac{1}{4}(N-2)}\left(\frac{\mu\omega}{\hbar}\xi^2\right)}{\Gamma(-\lambda/2-N/4+1)} + \frac{M_{\frac{\lambda}{2};-\frac{1}{4}(N-2)}\left(\frac{\mu\omega}{\hbar}\xi^2\right)}{\Gamma(-\lambda/2+N/4)} \qquad (13)$$

and $M_{\mu;\nu}(x)$ is thenormalized Whittaker's functionandwith $\xi^2+\xi'^2=\vec{r}^{\,2}+\vec{r}\,'^2$ and $\xi\xi' = \vec{r}.\vec{r}\,' = rr'\cos\theta$.

The expression (12) is a closed representation for the Green's function in terms of Whittaker's functions for the Schrödinger differential equation with a N-dimensional isotropic harmonic oscillator potential when N is one odd number.

For the even dimension it wasn't possible to obtain a closed representation in terms of Whittaker's functions. This problem was also discussed by Hostler[7] when he calculated the Coulomb Green's function in the N-dimensional space. He obtaines also a closed representation in terms of Whittaker's functions for the odd-dirnension but for evendimension only for the particular case where $\vec{r} \perp \vec{r}'$ (Ref.8).

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