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The Radial Coulomb Green's Function and Jacobi Green's Function

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The Jacobi Green's function is derived by means of the radial Coulomb Green's function.

Deriva-se a função de Green de Jacobi por meio da função de Green radial Colombiana.

1. INTRODUCTION

It's well known in quantum mechanics that the radial part of the Schrödinger equation for the Coulomb problem in momentum space is a particular Jacobi differential equation. In the present paper we use this fact to calculate the Jacobi Green's function.

2. JACOBI GREENS' FUNCTION

The Jacobi differential operator is

$$D_x = (1-x^2)\frac{d^2}{dx^2} - 2x\frac{d}{dx} - \left[\frac{(\beta+\alpha)^2+(\beta-\alpha)^2-2(\beta+\alpha)(\beta-\alpha)x}{1-x^2} - \nu(\nu+1)\right] \qquad (1)$$

where ν is integer, positive and the convenient choise $\beta+\alpha$, $\beta-\alpha$ for the

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parameters of the differential operator is only introduced to simplify the notation.

We first take out in D_x the weight function $(1-x)^{\alpha}\ (1+x)^{\beta}$ and derive the Green's function for the following differential operator

$$L_x = (1-x^2)\frac{d^2}{dx^2} + \left[2(\beta-\alpha) - 2(\beta+\alpha+1)x\right]\frac{d}{dx} + \mu(\mu + 2\alpha + 2\beta + 1) \tag{2}$$

where $\mu = \nu-\alpha-\beta$.

The Green's function for this differential operator satisfies the inhomogeneous differential equation

$$L_x g(x,x') = \delta(x-x') \tag{3}$$

which is bounded on $1 \leqslant x < \infty$; here $\alpha > -1$ and $\beta > -1$ in order to make the weight function non-negative and integrable but formal relations are valid without this restriction.

In order to obtain the corresponding differential equation in the momentum space we can use the fact that the Jacobi operator can be derived from the associated Legendre differential operator by derivation and we get the following differential equation

$$\left[\frac{d^2}{dp^2} - \frac{2\alpha+2\beta-2}{p}\frac{d}{dp} + 1 + \frac{2(\beta-\alpha)}{p}\,i - \frac{(\nu+\alpha+\beta)(\nu-\alpha-\beta-1)}{p^2}\right] F(p,p') =$$

$$= -\ (pp')^{\alpha+\beta-1}\ \delta(p-p') \tag{4}$$

This differential equation can be easily identified with the radial equation for a multidimensional Coulomb problem[1] and the solution is

$$F(p,p') = \frac{1}{2i}\ \frac{\Gamma(\nu+m+1)}{\Gamma(2\nu+2)}\ (pp')^{n-1}\ M_{-m;\nu+1/2}\ (2ip_<)\ W_{-m;\nu+1/2}\ (2ip_>) \tag{5}$$

where $n = \alpha+\beta$ and $m = \alpha-\beta$; $M_{\mu;\nu}(x)$ and $W_{\mu;\nu}(x)$ are the Whittaker functions and $p_<(p_>)$ is the lesser (greater) of p and p' respectively.

The Green's function for the L_x operator is calculated performing the Fourier anti-transformation of eq. (5) and it's not difficult to show that $g(x,x')$ is gived by[2]

$$g(x,x') = \frac{1}{4\pi i} \frac{\Gamma(\nu+m+1)}{\Gamma(2\nu+2)} \times$$

$$\times \int_0^\infty dp \int_0^\infty dp' \exp(-ipx)\, M_{-m;\nu+1/2}(2ip)\,(pp')^{n-1} \times$$

$$\times\; W_{-m;\nu+1/2}(2ip')\, \exp(ip'x') \tag{6}$$

In order to calculate this integral we use integrals representations for the Whittaker's functions[3] and we have for $g(x,x')$ the following integral.

$$g(x,x') = \frac{1}{2\pi} \frac{1}{\Gamma(\nu+m+1)\Gamma(\nu-m+1)}$$

$$\times \int_0^1 t'^{\nu+m}(1-t')^{\nu-m}\, dt' \int_0^\infty dp\; p^{\nu+n} \exp[-i(x+1-2t')p]$$

$$\times \int_0^\infty \exp(-t)\, t^{\nu+m}\, dt \int_0^\infty p'^{n-\nu-1}(2ip'+t)^{\nu-m} \exp[-i(1-x')p']\, dp' \tag{7}$$

with the restriction $\mathrm{Re}(\nu\pm m+1) \geqslant 0$.

The integral in the p variable gives

$$\exp\left[-\frac{i\pi}{2}(\nu+n+1)\right] \Gamma(\nu+n+1)\,(1+x-2t)^{-\nu-n-1}$$

and the integral in t'

$$2^{-\nu-n}\, Q_{\nu-n}^{(n+m;n-m)}(x)$$

where $Q_n^{(a,b)}(x)$ is the second Jacobi function. The integral in p' reproduces a Whittaker function[3]

$$(t)^{\frac{1}{2}(n-m-1)} \left(\frac{1-x'}{2}\right)^{\frac{1}{2}(m-n-1)} \exp\left|(1-x')t\right| \Gamma(n-\nu)\ W_{\frac{1}{2}(m-n+1);\frac{1}{2}(n-m)} \left(\frac{1-x'}{2}\ t\right)$$

and we obtain for $g(x,x')$ the integral

$$g(x,x') = -\ (2i)^{2n} \frac{\Gamma(\nu+n+1)\ \Gamma(n-\nu)}{\Gamma(\nu+m+1)\Gamma(\nu-m+1)} Q_{\nu-n}^{(n+m;n-m)}(x) \times$$

$$\times \left(\frac{1-x'}{2}\right)^{\frac{1}{2}(m-n-1)} \frac{1}{2\pi i} \int_{\infty(-\pi)}^{(0+)} dt\ \exp\left[-t-\frac{1-x'}{4}\ t\right] t^{\nu+\frac{n+m-1}{2}} \times$$

$$\times\ W_{\nu+\frac{1-n-m}{2};\frac{n-m}{2}} \left(\frac{1-x'}{2}\ t\right) \tag{8}$$

where we have made an analytic continuation in the t complex plane.

The calculation of this integral is straightforward and can be performed by means of hypergeometric functions and gives a $P_n^{(a,b)}(x)$ Jacobi function.

The final expression for $g(x,x')$ is

$$g(x,x') = 2^{-2(\alpha+\beta)} \frac{\Gamma(\nu+\alpha+\beta+1)}{\Gamma(\nu-\alpha+\beta+1)} \frac{\Gamma(\nu-\alpha-\beta+1)}{\Gamma(\nu+\alpha-\beta+1)}$$

$$\times P_{\nu-\alpha-\beta}^{(2\alpha,2\beta)}(x_<)\ Q_{\nu-\alpha-\beta}^{(2\alpha,2\beta)}(x_>) \tag{9}$$

The Green's function for the D_x operator is obtained incorporating the weight function in eq. (9).

$$G(x,x') = (1-x)^{\alpha}(1+x)^{\beta}(1-x')^{\alpha}(1+x')^{\beta} g(x,x')$$

3. CONCLUSIONS

Particular cases of the Jacobi Green's function are Legendre, enbauer and Tchebichef Green's function. If we put in eq.(9) $\alpha=\beta=m/2$,

m integer, we get the Green's function for associated Legendre function; $\alpha+\beta = \lambda-1/2$, $\alpha-\beta = 0$, with h > -1/2 for the Gegenbauer function. For the Tchebichef functions we have two special cases; $\alpha - \beta = 0, \alpha+\beta = -1/2$ for the first kind function and $\alpha-\beta = 0$, $\alpha+\beta = 1/2$ for the second kind function[4].

By means of this calculation of the Green's function we can also get integrals representation and various addition theorerns for this kind of Jacobi functions[5].

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