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Analysis of a Nonlinear Series RLC Circuit

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The problem of an electric circuit consisting of a resistor, an inductor, a capacitor and a couple of zenner diodes in series is studied in the framework of Convex Analysis. The differential inequality governing the circuit is shown to yield a unique stable solution which can be calculated through standard schemes. Numerical results are shown to agree with experiments.

O problema da resposta de um circuito elétrico, consistindo de uma resistência, uma bobina, um condensador e um par de diodos zenner em série, é estudado dentro do formalismo *da* Análise Convexa. Mostra-se que a inequação diferencial que governa o circuito possui uma solução estável que pode ser calculada através de algarismos usuais. Apresenta-se também o resultado de uma simulação digital do circuito, baseada em um algarismo tipo preditor-corretor, e compara-se este com medições tomadas em laboratório.

1. INTRODUCTION

In Ref. 1 we considered a couple of zenner diodes in parallel with the capacitor of a RLC circuit and showed, using concepts from Convex Analysis, that this configuration is the simplest one for the simulation of dielectric disruption from the phenomenological point of view. In the present paper we shall consider the zenner couple in series with the R, L and

C elements and analyse the differential inequality governing the circuit with the same methodology.

We point out firstly that the elasto-plastic spring is the mechanical analogue to the parallel configuration, as well as the elastic spring subjected to a dry friction is the analogue to the in series configuration of the circuit. A similar mathematical analysis to the one we shall develop here can be seen, for example, in references 2,3, where a model for a pile-soil interaction through friction is discussed. Secondly, we remark that those two basic possibilities to arrange the couple of zenner diodes with the capacitor in the circuit (parallel and series) correspond to the two possibilities of having variational inequalities to represent a system: to have unilateral constraints on some variable or to deal with non-differentiable functionals.

The aim of this article is to present theoretical, computational and experimental results about the nonlinear series RLC circuit. The plan of it is the following. In Section 2 the physical model is described. The functional framework employed and a theorem asserting that we have a mathematically well posed problem are explained in Section 3. Section 4 contains a description of the computing algorithm and the related convergente results. In Section 5 the output of a numerical simulation is presented, and then compared with laboratory measurements in the next Section 6. All the mathematical proofs are developed in Section 7. Finally, in Section 8, we summarize general conclusions.

2. THE PHYSICAL MODEL

The fundamental circuit equations are

- Conservation laws (Kirchoff):

$$E(t) = V_L + V_R + V_Z + V_C \ , \qquad (2.1)$$

$$I(t) = I_L = I_R = I_Z = I_C = \frac{dq_C}{dt} \ ,$$

(see Figure 1),

where

$E(t)$ = applied voltage,

q_c = charge at the capacitor's plate,

(V_L, I_L) = (voltage, current) at the inductor,

(V_R, I_R) = (voltage, current) at the resistor,

(V_Z, I_Z) = (voltage, current) at the cell representing the couple of zenner diodes,

(V_C, I_C) = (voltage, current) at the capacitor;

- Behavior laws:

$$V_L(t) = L \frac{dI(t)}{dt} \quad \text{(Lenz)} ,$$

$$V_R(t) = R\, I(t) \quad \text{(Ohm)} ,$$

$$q_c(t) = C\, V_C(t) , \tag{2.2}$$

$$I_Z(t) = F(V_Z(t)) ,$$

where F, ths characteristic function of the arrangement of zenners, is given by the graph shown in Figure 2. All the physical parameters R, L, C and Z are assumed to be constants, and positives.

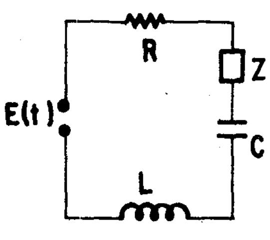

Figure 1

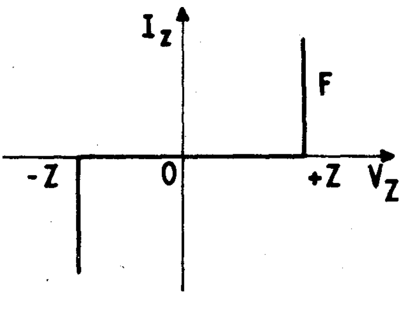

Figure 2

The graph F is "monotonic", hence can be inverted, and (see Figure 3) its inverse

$$V_Z = F^{-1}(I_Z) = \begin{cases} \{+Z\} & \text{if } I_Z > 0 , \\ \{-Z,Z\} & \text{if } I_Z = 0 , \\ \{-Z\} & \text{if } I_Z < 0 , \end{cases} \qquad (2.3)$$

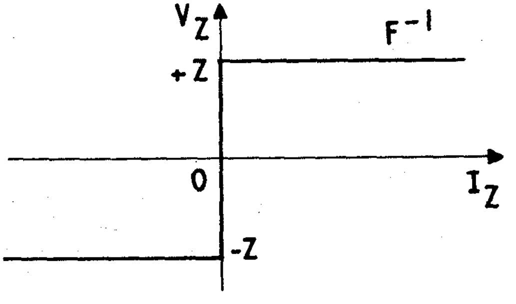

Figure 3

is another graph.

Actually, the physical obtention of such characteristic response from the nonlinear element involves the use of an adequate high gain and low output impedance operational amplifier together with the two diodes, as for example the one shown in Figure 4.

Adding up, the state of the circuit is described by the initial value problem, for $I(t)$ and $q_c(t)$,

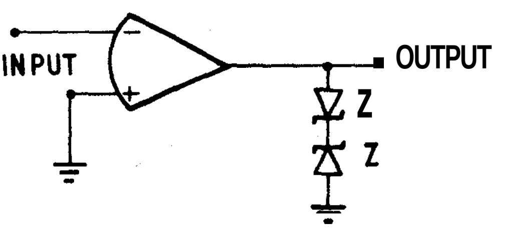

Figure 4

$$\text{(i)} \quad q_c(0) = 0 , \quad I(0) = 0 ,$$

$$\text{(ii)} \quad \frac{dq_c}{dt} = I(t) \quad , \quad t > 0 , \qquad (2.4)$$

$$\text{(iii)} \quad L \ \text{dt} + \text{R I }) + \frac{1}{C} q_c(t) + F^{-1}(I(t)) = \text{E }) \quad , t > 0 ,$$

where equation (2.4) (i ii) is to be correctly interpreted.

3. THE MATHEMATICAL SET-UP

The correct interpretation ot (2.4) is the following. We consider the convex function

$$f(x) = Z|x| \quad , \quad x \in R \quad , \qquad (3.1)$$

and the definition of the sub-differential $\partial f(x)$ of a convex function f at a point $x \in$ R; a set defined by the criterion

$$5 \in \partial f(x) \quad \text{if and only if } f(y) \geqslant f(x) + \zeta(y-x) \ ,$$

$$\forall y \in R \qquad (3.2)$$

We shall have then

$$\partial f(x) = F^{-1}(x) \quad , \quad \text{x} \in \text{R} \quad ,$$

and the following formulation of (2.4) (iii):

$$\left[E - L \frac{dI}{dt} - \text{RI} - \frac{1}{C} q_c \right](t) \in \partial f(I(t)) \ , \quad \forall t > 0 \ . \qquad (3.3)$$

In view of definition (3.2), an alternative formulation to (3.3) is

$$\left[L \frac{dI}{dt} + RI + \frac{1}{C} q_c \right](t) \ \left[j - I(t)\right] + f(j) - f(I(t)) \geqslant$$

$$E(t) \ \left[j - I(t)\right] \quad , \quad \forall j \in R , \quad \forall t > 0 \ . \qquad (3.4)$$

Hence, eliminating the variable $q_{C'}$ the combination of equations (2.4) and (3.4) results in the following initial value problem for the description of the circuit:

to find ■ $= I(t), \quad t \geqslant 0$, such that

(i) $I(0) = 0$,

(ii) $\left[L \dfrac{dI}{dt} + RI + \dfrac{1}{C} \displaystyle\int_0^t I(\tau)\, d\tau\right]\left[j - I(t)\right] + f(j) -$

$- f(I(t)) \geqslant E(t) \left[j - I(t)\right] , \forall j \in R, \quad \forall t > 0.$

(3.5)

Remark

The classical series RLC circuit correqponds to the limit case $Z=0$. If then we take in (3.5) (ii) $j = I(t) \pm 1$ we obtain the integro-differential equation

$$L \frac{dI}{dt} + RI + \frac{1}{C} \int_0^t I(\tau)\, d\tau = E(t) \ , \ t > 0 \ .$$

Let us introduce now some functional space which we shall make use for the precise mathematical description of the results obtained. If $0 < T < \infty$ let

$C^0(0,T)$ = space of continuous functions $v(t)$ on $[0,T]$,

with norm $|v|_0 = \max |v(t)| \ ; \quad 0 \leqslant t \leqslant T$

$L^p(0,T)$ = space of functions $v(t)$ on $[0,T]$ for which $|v(t)|^p$

is Lebesgue integrable, with norm

$$|v|_p = \left[\int_0^T |v(t)|^p \, dt \right]^{1/p} \ , \quad p \geqslant 1 \ ;$$

L $(0,T)$ = space of all Lebesgue measurable functions $v(t)$ on

$[0,T]$ which are bounded, except possibly on a set

of measure zero, with norm

$$|v|_\infty = \inf_{u(t)=v(t)\ \text{a.e.}} \{ \sup_{0 \leq t \leq T} |u(t)| \} .$$

The theoretical analysis of the circuit which we present is summarized in the following.

Theerem 3.1

Let $0<T\leq\infty$ and $E \in L^2(0,T)$ be given such that $dE/dt \in L^2(0,T)$. Then there exists a unique function $I=I(t)$, satisfying

$$I \in L^\infty(0,T) \quad L^2(0,T) \quad C^0(0,T) , \tag{3.6}$$

$$\frac{dI}{dt} \in L^\infty(0,T) \quad L^2(0,T) , \tag{3.7}$$

$$q_c(t) = \int_0^t I(\tau)d\tau , \quad q_c \in L^\infty(0,T) \tag{3.8}$$

which verifies (3.5) for almost every $t \in (0,T]$. Furthermore, the solution map $E(t) \xrightarrow{F} I(t)$ is continuous, in the sence that there exists a constant K such that

$$|I^1-I^2|_\infty+|I^1-I^2|_2+|q_c^1-q_c^2|_\infty \leq K|E^1-E^2|_2 , \tag{3.9}$$

where $I^i = F(E^i)$, $i=1,2$, and $q_c^i(t) = \int_0^t I^i(\tau)d\tau$. And when $T \to \infty$, we have $I(T) \to 0$.

This theorem essentially guarantees that we are dealing whita well posed nathematical problem, that is, the internal consistency of our mathematical model for the circuit.

4. THE COMPUTING ALGORITHM

To compute approximate solutions of (3.5) we can use a method based on a convex regularization of the function $f(x)$ and a standard discretization in time for the resulting ordinary differential equation.

We take, for $\varepsilon > 0$,

$$f_\varepsilon(x) = \begin{cases} Z(|x| - \frac{\varepsilon}{3}) & \text{if } |x| \geq \varepsilon , \\ Z\varepsilon \left[(\frac{x}{\varepsilon})^2 - \left|\frac{x}{\varepsilon}\right|^3 /3 \right] & \text{if } |x| \leq \varepsilon . \end{cases} \tag{4.1}$$

Then we modify problem (3.5) substituting $f(x)$ by $f_\varepsilon(x)$. Since the latter is differentiable, the variational inequality is substituted by the ordinary differential equation

$$L \frac{dI_\varepsilon(t)}{dt} + RI_\varepsilon(t) + \frac{1}{C} q_c^\varepsilon(t) + f'_\varepsilon(I_\varepsilon(t)) = E(t), \tag{4.2}$$

with

$$f'_\varepsilon(t) = \begin{cases} Z \operatorname{sgn}(x) & , \text{ if } |x| \geq \varepsilon , \\ 2 \frac{x}{\varepsilon} Z - (\frac{x}{\varepsilon})^2 Z \operatorname{sgn}(x) & , \text{ if } |x| \leq \varepsilon . \end{cases}$$

Now for (4.2) and (2.4) (ii) the time discretization proposal is a Crank-Nicolson scheme. Given $T>0$ finite and fixed once for all, we introduce an arbitrary positive integer N and

$$H = T/N \quad , \quad t_n = nH \quad , \quad n = 0,1,\dots,N$$

After this we consider the basic formula

$$\begin{aligned} &(\text{i}) \quad I^0 = 0, \; Q_c^0 = 0 , \\ &(\text{ii}) \quad L\, \partial_t I^n + RI^{n+1/2} + \frac{1}{C} Q_c^{n+1/2} = E^{n+1/2} - f'_\varepsilon(I^{n+1/2}), \\ &\qquad n = 0,1,\dots,N-1 , \\ &(\text{iii}) \quad \partial_t Q_c^n = I^{n+1/2} \quad , \quad n = 0,1,\dots,N-1 , \end{aligned} \tag{4.3}$$

where $\{I^n, Q_c^n\}_{n=0}^N$ intends to be an approximation to $\{I(t_n), \; q_c^\varepsilon(t_n)\}_{n=0}^N$, and

$$\partial_t G^n = H^{-1}(G^{n+1} - G^n) ,$$

$$G^{n+1/2} = 2^{-1}(G^{n+1} + G^n) ,$$

for any sequence $\{G^n\}_{n=0}^N$.

The computing algorithm associated with (4.3) is thefollowing predictor cor'rector version of it:

(S_0) I^0, $Q_c^0 = 0$;

$(\tilde{S}_n)$ I^n, Q_c^n known, compute predictors $\hat{I}^{n+1}$, $\hat{Q}_c^{n+1}$ in (4.3) (ii) and (iii) with:

$\{\hat{I}^{n+1}, \hat{Q}_c^{n+1}\} \to \{I^{n+1}, Q_c^{n+1}\}$ in the left hand side of (4.3) (ii) and in (4.3) (iii);

$\{I^N\} \to \{I^{n+1}\}$ in the right hand side of (4.3) (ii);

(S_n) $I^n, Q_c^n, \hat{I}^{n+1}, \hat{Q}_c^{n+1}$ known, compute correctors I^{n+1}, Q_c^{n+1}, in (4.3) (ii) and (iii) with $\hat{I}^{n+1} \to I^{n+1}$ in the right hand side of (4.3)(ii);

(S_N) stop at $n = N-1$. (4.4)

Algorithm (4.4) is unconditionally convergent with $H \to 0$, and $\varepsilon \to 0$. We shall use it in the numerical simulation to be presented next Section. In more precise terms, we can state the following result, the proof of which will be posponed to Section 7.

Theorem 4.1

Under the hypothesis of Theorem 3.1, the following equalities hold for the iterated limits:

$$\lim_{\varepsilon\to 0} \lim_{H\to 0} \sup_{0\leq n\leq N} |I^n - I(t_n)| = 0 , \tag{4.5}$$

$$\lim_{\varepsilon\to 0} \lim_{H\to 0} \sup_{0\leq n\leq N} |Q_c^n - q_c(t_n)| = 0 , \tag{4.6}$$

$\{I,q_c\}$ being the solution of (2.4) (i), (2.4)(ii), (3.4) and $\{I^n, Q_c^n\}$ being defined in (4.4).

5. A NUMERICAL SIMULATION

In this Section we present a typical numerical resultobtained by implementing a algorithm (4.4) as a Fortran program. This program was run in the IBM 370/145 machine at LCC-CNPq, in double precision. For the input

$$L = 10^{-2} \ , \quad C = 10 \ , \quad R = 1 \ ,$$

$$Z = 4 \times 10^{-1} \ , \quad \varepsilon = 10^{-7} \ , \quad H = 10^{-2} \ ,$$

$$T = 2 \times 10 \ ,$$

$$E(t) = \begin{cases} 2 & \text{if } t \in [jH, (j+1)H] \ , \ j \text{ even}, \\ 0 & \text{if } t \in (jH, (j+1)H) \ , \ j \text{ odd} \ , \end{cases}$$

the output described by the two curves of Figure 5 was obtained. The curve $Q_c = Q(t)$ indicates the process of charge storage at the capacitor, the reservoir of electrical energy in the circuit.

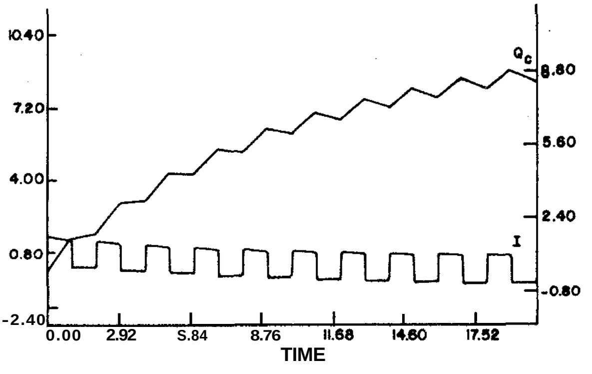

Figure 5

6. COMPARISON WITH EXPERIMENTAL RESULTS

The experimental verification of the theoretical numerical results presented in the previous sections was made using the non-linear properties of' an element consisting of two zenner diodes and an operational amplifier as shown earlier in Figure 4. As mentioned before, this amplifier (μA741) has a very low output impedance and a high gain. The diodes used in the experiment have the zenner voltage of 4.7 volts.

The complete circuity is shown in Figure 6. The part of the circuit which is attached to the capacitor has the function of discharging it whenever its voltage reaches a certain value, fixed by the voltage V_1. This voltage, on the other hand, must be adjusted in orderto produce a repetitive picture in the osciloscope screen. In case V_1 is too high one can observe only the permanent regime of the circuit.

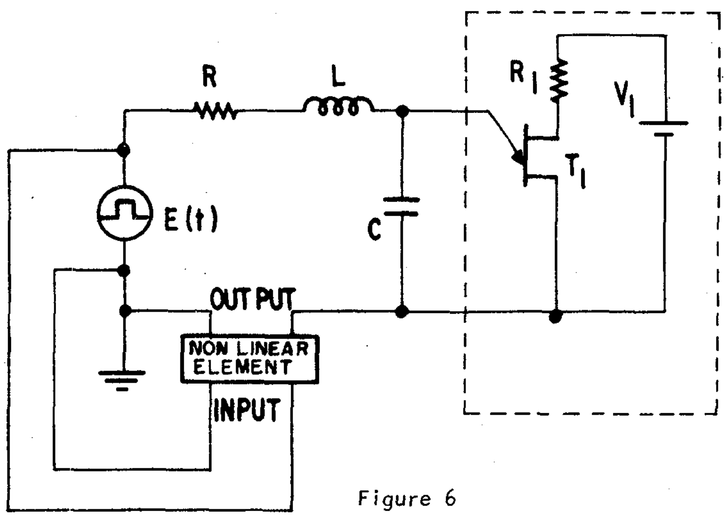

Figure 6

Figure 7 shows at the top the measured voltage at the capacitor's plates, as function of time. Since the charge is linearly related to this voltage, it also indicates the way the capacitor is charged under the applied voltage $E(t)$, which in the present case is a train of rectangular pulses, as shown in the bottom of Figure 7.

As it can be clearly seen, the amount of additionalcharge which is added to the capacitor by each individual pulse decreaçes with the time. This result qualitatively agrees with the numerical prediction (see Figure 5) and also with the simulation for the equivalent rnechanical system considered in Ref.3.

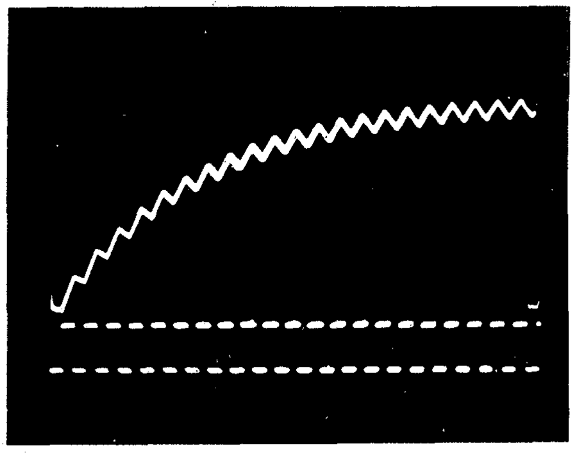

Figure 7

The values used in this case were: R=3KΩ, L=10mH, C=1μF. The scales are such that each horizontal division corresponds to 1μs and each vertical division to 20 volts in the input and 2 volts in the output. The elements of the auxiliary circuit were: $V_1 \cong 3$V, R_1=560Ω, and T_1 being the UJT 2N2646 component.

7. MATHEMATICAL PROOFS

Technical details of the proofs of the assertions made in Theorems 3.1 and 4.1 will be discussed now.

7.1. Proof of Theorem 3.1

The proof consists in a sequence of claims and corresponding justifications.

Claim 1 (Uniqueness)

If solutions of (3.5) exists, they cannot be more than just one.

Assume there are two solutions, I^1 and I^2. We have

$$\left[L\frac{dI^1}{dt}+RI^1+\frac{1}{C}\int_0^t I^1(\tau)d\right]\left[j-I^1(t)\right]+f(j)-$$

$$-f(I^1(t))\geq E\left[j-I^1(t)\right],\ \forall j\in R,\ \forall t>0, \qquad (7.1)$$

$$\left[L\frac{dI^2}{dt}+RI^2+\frac{1}{C}\int_0^t I^2(\tau)d\tau\right]\left[j-I^2(t)\right]+f(j)-$$

$$-f(I^2(t))\geq E\left[j-I^2(t)\right],\ \forall j\in R,\ \forall t>0, \qquad (7.2)$$

$$I^1(0)=I^2(0)=0\ . \qquad (7.3)$$

Taking $j=I^2$ in (7.1), $j=I'$ in (7.2), and adding,

$$I(t)\left[-L\frac{dI}{dt}-RI-\frac{1}{C}\int_0^t I(\tau)d\tau\right]\geq 0\ ,$$

where $I=I^1-I^2$. Hence

$$\text{(i)}\quad I(0)=0\ ,$$

$$\text{(ii)}\quad \frac{L}{2}\frac{d}{dt}\left[I(t)\right]^2+R\left[I(t)\right]^2+\frac{1}{2C}\frac{d}{dt}\left[\int_0^t I(\tau)d\tau\right]^2\leq 0\ ,$$

$$t>0\ . \qquad (7.4)$$

By integration on time *(7.4)* implies

$$\frac{L}{2}\left[I(t)\right]^2+R\int_0^t\left[I(\tau)\right]^2d\tau+\frac{1}{2C}\left[\int_0^t I(\tau)d\tau\right]^2\leq 0\ ,\ t>0$$

That is, $I(t)=0,\ t\geq 0$, and $I^1(t)=I^2(t),\ t\geq 0$. The two solutions must coincide, and the claim is justified.

Claim 2 (Convex regularization of problem (3.5))

Given $\varepsilon>0$, the initial value problem

$$\text{(i)}\quad I_\varepsilon\in L^\infty(0,T)\cap L^2(0,T)\cap C^0(0,T)\ ,$$

$$\text{(ii)}\quad \frac{dI_\varepsilon}{dt}\in L^\infty(0,T)\cap L^2(0,T)\ ,$$

(iii) $q_C^\varepsilon \in L^\infty(0,T) \cap C^0(0,T)$,

(iv) $\frac{dq_C^\varepsilon}{dt}(t) = I_\varepsilon(t)$, $0 < t < T$,

(v) $L\frac{dI_\varepsilon}{dt} + RI_\varepsilon + \frac{1}{C}q_C^\varepsilon + f'_\varepsilon(I_\varepsilon(t)) = E(t)$, $0 < t < T$,

(vi) $I_\varepsilon(0) = 0$, $q_C^\varepsilon(0) = 0$,

(7.5)

where f_ε is defined in (4.1), well defines the sequences $\{I_\varepsilon\}_{\varepsilon > 0}$ and $\{q_C^\varepsilon\}_{\varepsilon > 0}$.

Being the associated function to system (7.5) (iv) and (v) continuous and monotone, such a system has a unique local solution which can be extended to $[0,T]$ in case $I_\varepsilon(t)$ and $q_C^\varepsilon(t)$ are bounded on the interval. In the sequel we shall prove two a priori estimates for (7.5) (iv) and (v) which will fulfill this boundedness requirements as well as conditions (7.5)(i), (ii) and (iii), so demonstrating claim 2.

For the first estimate (energy) we multiply (7.5)(v) by $I_\varepsilon(t)$, getting

$$\frac{L}{2}\frac{d}{dt}I_\varepsilon^2(t) + RI_\varepsilon^2(t) + \frac{1}{2C}\frac{d}{dt}\left[q_C^\varepsilon(t)\right]^2 + f'_\varepsilon(I_\varepsilon(t))I_\varepsilon(t) = E(t)I_\varepsilon(t) .$$

Integrating on time and using the initial conditions:

$$\frac{L}{2}I_\varepsilon^2(t) + R\int_0^t I_\varepsilon^2(\tau)d\tau + \frac{1}{2C}\left[q_C^\varepsilon(t)\right]^2 + \\ + \int_0^t f'_\varepsilon(I_\varepsilon(\tau))I_\varepsilon(\tau)d\tau = \int_0^t E(\tau)I_\varepsilon(\tau)d\tau ,$$

which is the energy identity for our system. Hence, since

$$f'_\varepsilon(x)x \geq 0 \quad , \quad x \in R$$

$$\frac{L}{2}|I_\varepsilon|_\infty^2 + \frac{R}{2}|I_\varepsilon|_2^2 + \frac{1}{2C}|q_C^\varepsilon|_\infty^2 \leq \frac{1}{2R}|E|_2^2 \qquad (7.6)$$

For the second estimate we start differentiating equation (7.5) (v):

$$L \frac{d^2 I_\varepsilon}{dt^2} + R \frac{dI_\varepsilon}{dt} + \frac{1}{C} I_\varepsilon + \lim_{\Delta t \to 0} \frac{f'_\varepsilon(I_\varepsilon(t+\Delta t)) - f'_\varepsilon(I_\varepsilon(t))}{\Delta t} = E'(t), \quad 0<t<T .$$

Now we multiply it by I'_ε ,

$$\frac{L}{2} \frac{d}{dt} \left[\frac{dI_\varepsilon}{dt}\right]^2 + R \left[\frac{dI_\varepsilon}{dt}\right]^2 + \frac{1}{2C} \frac{d}{dt} I_\varepsilon^2 +$$

$$+ \lim_{\Delta t \to 0} \frac{\left[f'_\varepsilon(I_\varepsilon(t+\Delta t)) - f'_\varepsilon(I_\varepsilon(t))\right]\left[I_\varepsilon(t+\Delta t) - I_\varepsilon(t)\right]}{[\Delta t]^2}$$

$$= E'(t)\, I'_\varepsilon(t) ,$$

observe that

$$\left[f'_\varepsilon(I_\varepsilon(t+\Delta t)) - f'_\varepsilon(I_\varepsilon(t))\right]\left[I_\varepsilon(t+\Delta t) - I_\varepsilon(t)\right] \geqslant 0 ,$$

and integrate on time to obtain

$$\frac{L}{2} \left(\frac{dI_\varepsilon}{dt}\right)^2 + R \int_0^t \left[\frac{dI_\varepsilon}{dt}\right]^2 (\tau)\, d\tau + \frac{1}{2C} I_\varepsilon^2(t) \leqslant$$

$$\leqslant \frac{L}{2} \left[\frac{dI_\varepsilon}{dt}\right]^2 (0) + \int_0^t E'(\tau) I'_\varepsilon(\tau)\, d\tau .$$

if we look equation (7.5) (v) at $t=0$ we get

$$L \frac{dI_\varepsilon}{dt} (0) = E(0).$$

Hence we obtain the a priori estimate

$$\frac{L}{2} \left|\frac{dI_\varepsilon}{dt}\right|_\infty^2 + \frac{R}{2} \left|\frac{dI_\varepsilon}{dt}\right|_2^2 + \frac{1}{2C} |I_\varepsilon|_\infty^2 \leqslant \frac{|E(0)|^2}{2L} + |E'|_2^2 . \qquad (7.7)$$

Since the right hand sides of (7.6) and (7.7) involve only data, and are independent of ε, we can conclude that

(i) q_c^{ε} are all in a bounded set of $L^{\infty}(0,T)$,

(ii) I_{ε} are all in a bounded set of $L^{\infty}(0,T) \cap L^2(0,T)$,

(iii) $\frac{dI_{\varepsilon}}{dt}$ are all in a bounded set of $L^{\infty}(0,T) \cap L^2(0,T)$.

This implies in particular (7.5)(ii) and part of (7.5)(i) and (iii). Now, q_c^{ε} and I_{ε} are continuous because of

$$|q_c^{\varepsilon}(t) - q_c^{\varepsilon}(\tau)| = \left| \int_{\tau}^{t} I_{\varepsilon}(s)ds \right| \leq |I_{\varepsilon}|_{\infty} |t-\tau| \leq$$

$$\leq \text{const.} \ |t-\tau| , \tag{7.8}$$

$$|I_{\varepsilon}(t) - I_{\varepsilon}(\tau)| = \left| \int_{\tau}^{t} \frac{dI_{\varepsilon}}{ds}(s)ds \right| \leq \left| \frac{dI_{\varepsilon}}{dt} \right|_{\infty} |t-\tau| \leq$$

$$\leq \text{const.} \ |t-\tau| , \tag{7.9}$$

respectively.

The proof of claim 2 is ended

Claim 3 (Existence)

There exists a solution I of (3.5)-(3.8).

Estimates (7.6), (7.8) and (7.9) imply that the sets $\{q_c^{\varepsilon}\}_{\varepsilon>0}$ and $\{I_{\varepsilon}\}_{\varepsilon>0}$ are uniformly bounded and equicontinuous. Hence by Arzela's theorem there exists a limit point $\{q_C, I\}$ such that $q_c^{\varepsilon} \to q_C$ and $I_{\varepsilon} \to I$, uniformly on $[0,T]$, for some subsequence. We shall show that this limit point is the desired solution.

In fact, $dq_C/dt = I$ and by the conclusions (i), (ii) and (iii) of (7.6) and (7.7) we have that I and q_C satisfy (3.6)-(3.8). We need then only to show that I satisfies (3.5) (ii) for almost every $t \in [0,T]$.

Equation (7.5)(v) and the fact that f_ε is a convex function imply

$$f_\varepsilon(j(t)) - f_\varepsilon(I_\varepsilon(t)) - \left|E(t) - L\frac{dI_\varepsilon}{dt} - RI_\varepsilon - \right.$$

$$\left. - \frac{1}{C}\, q_c^\varepsilon\right| |j(t) - I_\varepsilon(t)| \geqslant 0 \ , \ \forall j \in L^1(0,T) \ ,$$

$$0 < t < T \ .$$

Integrating from 0 to T :

$$L\int_0^T \frac{dI_\varepsilon}{dt}\, j\, dt + R\int_0^T I_\varepsilon\, j\, dt + \frac{1}{C}\int_0^T q_c^\varepsilon\, j\, dt + \int_0^T f_\varepsilon(j)dt \geqslant$$

$$\geqslant \int_0^T E|j-I_\varepsilon|dt + \int_0^T f_\varepsilon(I_\varepsilon)dt + \frac{L}{2}I_\varepsilon^2(T) + R\int_0^T I_\varepsilon^2\, dt +$$

$$+ \frac{1}{2C}\,|q_c^\varepsilon(T)|^2 \ , \ \forall j \in L^1(0,T) \ . \qquad (7.10)$$

But we know from the compactness properties of bounded sets in L^p spaces that we can ertract from $\{I_\varepsilon, q_c^\varepsilon\}_{\varepsilon>0}$ a subsequence, still denoted by $\{I_\varepsilon, q_c^\varepsilon\}_{\varepsilon>0}$, such that

$$\begin{aligned} q_c^\varepsilon &\to q_c \quad \text{uniformiy on } [0,T] \ , \\ I_\varepsilon &\to I \quad \text{uniformly on } [0,T] \ , \\ \frac{dI_\varepsilon}{dt} &\to \frac{dI}{dt} \quad \text{weakly * in } L^\infty(0,T) \ . \end{aligned} \qquad (7.11)$$

Hence, if we take the limit $\varepsilon\downarrow 0$, *(7.10)* and (7.11) yield

$$L\int_0^T \frac{dI}{dt}\, j\, dt + R\int_0^T I(j)dt + \frac{1}{C}\int_0^T q_c\, j\, dt + \int_0^T f(j)dt \geqslant$$

$$\geqslant \int_0^T E[j-I]dt + \int_0^T f(I)dt + \frac{L}{2}I^2(T) + R\int_0^T I^2 dt +$$

$$\frac{1}{2C}\,[q_c(T)]^2 \ , \ \ \forall j \in L^1(0,T) \ ,$$

that is,

$$\int_0^T \left[L \frac{dI}{dt} + RI + \frac{q_c}{C}\right] [j-I]dt + \int_0^T [f(j)-f(I)\ dt] \geqslant$$

$$\geqslant \int_0^T E(j-I)dt\ , \quad \forall j \in L^1(0,T)\ . \tag{7.12}$$

Finally, through a standard measure-theoretic argument (see the existence proof in Reé.2) one can show that (7.12) is equivalent to *(3.5)* (ii), except possibly in a set of measure zero in $[0,T]$.

Claim 4 (Stability)

We have that (3.9) holds true and $I(t) \to 0$ as $t \to \infty$.

By definition:

$$\left[L \frac{dI^\alpha}{dt} + RI^\alpha + \frac{1}{C} q_c^\alpha\right] [j-I^\alpha] + f(j) - f(I^\alpha) \geqslant$$

$$\geqslant E^\alpha(t)\ [j-I^\alpha(t)]\ , \ \alpha = 1,2\ , \quad \forall j \in R\ , \quad \forall t > 0\ ,$$

$$I^\alpha(0) = 0\ , \quad \alpha = 1,2\ .$$

Taking $j = I^2$ in equations $\alpha = 1$ and $j = I^1$ in equations a = 2, adding up and integrating from 0 to t:

$$\frac{L}{2}(I^1-I^2)^2(t) + R\int_0^t (I^1-I^2)^2(\tau)d\tau + \frac{1}{2C}(q_c^1-q_c^2)^2(t) \leqslant$$

$$\leqslant \int_0^t (E^1-E^2)(I^1-I^2)dt \leqslant \frac{R}{2}\int_0^t (I^1-I^2)^2(\tau)d\tau +$$

$$+ \frac{1}{2R}\int_0^t (E^1-E^2)^2(\tau)d\tau.$$

This implies (3.9).

Now take $T=\infty$. Since $I(t)$ is uniformly continuous on $[0, +\infty)$ (see **(7.9)**) and also square-integrable, then $I(t) \to 0$ as $t \to \infty$. And the proof of theorem 3.1 is ended.

7.2. Proof of Theorem 4.1

F'or the sake of space saving we do not present this proof here. We refer the reader to Ref.3. There a proof is developed for a formally more general situation. An argument for the present case can be build following step by step the proofs of Lemma 1 and Theorem 2, in Section 6, and making the appropriate correspondences and reductions.

8. CONCLUSIONS

A mathematical model (equation (3.5)) is proposed to analyze the response of an electric circuit consisting of elements of resistance, inductance, capacitance and a non-linearity described by Figure 2, all in series, to an applied voltage $E(t)$. The well-posedness of this response problen is demonstrated, both for the continuous and discretized versions of the model (theorems 3.1 and 4.1). A model consistent numerical calculation was performed in the IBM 370/145 computer at the Laboratório de Computação Científica - CNPq and an electric charge x time curve was obtained which agreed with experimental measurements held at the Laboratório de Eletrônica of the Universidade de Brasília.

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