Revista Brasileira de Física, Vol. 11, nº 1, 1981

Three-Alpha Forces in the ^{16}O Nucleus*

D. A. AGRELLO**, V. C. AGUILERA NAVARRO and J. N. MAKI**
Instituto de Física Teórica, Rua Pamplona 145, CEP 01405 – São Paulo, Brasil

Recebido um 22 de Agosto de 1980

The effect of recently determined three-alpha forces in the ground-state energy and elastic form factor of ^{16}O is discussed, in the alpha-particle model.

Discutem-se os efeitos de forças de três alfas, recentemente propostas na literatura, na energia e no fator de forma elástico do estado fundamental do ^{16}O, no modelo de partícula alfa.

1. INTRODUCTION

In this paper, we shall discuss the effect of recently determined three-alpha forces[1,2] in the ground state of the ^{16}O nucleus. Due to the nature of those forces, and of the two-body interaction here considered, the oxigen nucleus is properly described in the alpha-particle model. In this very simplified description, the four alpha particles are considered as structureless and indestructible entities, each with electric charge equal to $2e$. Although these partirles can be taken as point ones for the discussion of the spectrum, a correction due to its extension in space is taken into account in the study ofthe form factor of the nucleus.

Previous calculations using the aboue alpha-particle model and two-alpha forces have failed in satisfactorily describingtheground

* Partially supported by FINEP, Brasil, under contract 522/CT.
** CNPq fellowship, Brasil.

-state energy of the ^{16}O nucleus, in spite of succeeding in the description of other nuclear properties as the radius and main aspects of the charge form factor, for example. If one wants to exploit a bit more the model, it is clearly of interest to invastigate the effect of a three-alpha force acting among the nucleus constituents. In the following, we develop this program.

The mathematical method used is a variational analysisofthe hamiltonian. First we consider a hamiltonian constructed of only two-body forces - plus the total kinetic energy, obviously. Next, we turn on the three-alpha interaction arid repeat the calcularions to compare results. Indeed, we take advantage of some results by other authors[3,4] concerning the calculations with two-alpha forces only.

The trial function is expanded in terms of translationally invariant four-particle harmonic-oscillator functions. It is constructed in such way as to have definite total orbital angular momentum L-0 arid positive parity. The variational function is completely symmetric as it should be if we want it to describe a boson system. The explicit construction of this function is discussed in the next section.

The presentation and discussion of the three-alpha forces considered are made iri section 3, where the matrix elements of the hamiltonian are also discussed. The form factor is calculated in section 4 and all the results of this paper are discussed in the last section.

2. THE VARIATIONAL FUNCTION

The ground state of the ^{16}O nucleus, described in terms of four structureless and indestructible alpha particles, is represented by a completely symmetric wavefunction with total orbital angular momentum L=0 and positive parity. Since the forces considered are spin independent, the variational anaiysis we intend to carry out can be done entirely in the configuration space so that the above characterization of the wavefunction will be enough. In order to avoid spurious energies, the wavefunction must be in addition translationally invariant. In the following, we shall construct four-particle harmonic- os-

cillator states with the above properties and then expand the variational function in terms of such states. Since this was already done in some details elsewhere[4,5], we now restric ourselves to a brief presentation of that matter.

The trial function is expressed as the expansion

$$\psi = \sum_{\nu} a_{\nu} \phi_{\nu} , \tag{2.1}$$

where a_{ν} are variational parameters to be determined and ϕ_{ν} are four-particle harmonic-oscillator wavefunctions (FPHO). The index v stands for the FPHO quantum numbers. The cut in the sum (2.1) is determined by the degree of the approximation desired (or allowed by the costs involved). This is defined by the maximum quantum number of the FPHO components.

It is very convenient to introduce the Jacobi coordinates for a system of four particles, namely,

$$\vec{x}_a = \sqrt{1/2}\ (\vec{x}_1 - \vec{x}_2) , \tag{2.2a}$$

$$\vec{x}_b = \sqrt{1/6}\ (\vec{x}_1 + \vec{x}_2 - 2\vec{x}_3) \tag{2.2b}$$

$$\vec{x}_c = \sqrt{1/12}\ (\vec{x}_1 + \vec{x}_2 + \vec{x}_3 - 3\vec{x}_4) \tag{2.2c}$$

$$\vec{x}_d = 1/2\ (\vec{x}_1 + \vec{x}_2 + \vec{x}_3 + \vec{x}_4), \tag{2.2d}$$

where $\vec{x}_{\alpha}$, a = a,b,c,d, are the coordinates of the four particles referred to an arbitrary frame. These coordinates are given in inities of $(\hbar/m\omega)^{1/2}$, where m is the alpha-particle mass and ω the oscillator frequence. It must be noted that $\vec{x}_d$ is essentially the center of mass coordinate. The other $\vec{x}$ are-translationally invariants. As it will be shown later on, the coordinate $\vec{x}_a$ is usefull in the calculation of two-body matrix elements, while $\vec{x}_b$ is adequate for the evaluation of three-body matrix elements. The form factor depends on the coordinate $\vec{x}_c$. These comments largely justify the choice of the coordinates (2.2).

Let $|n_{\alpha} \ell_{\alpha} m_{\alpha}>$ stand for the state of a harmonic oscillator in

the coordinate $\vec{x}_\alpha$. If we allow only zero-quantum excitation in the center of mass coordinate, the total number of quanta N associated to four oscillators in the coordinates (2.2) is given by

$$N = 2n_a+R_a+2n_b+R_b+2n_c+R_c \quad , \tag{2.3}$$

for $n_d = \ell_d = 0$.

The FPHO state with definite total orbital angular momentum L is obtained from the $|n_a \ell_a m_a>$ by ordinary coupling. The particular case **L=0** will be denoted simply by

$$|n_a\ell_a,n_b\ell_b,n_c\ell_c> \tag{2.4}$$

It can be easily shown that the state (2.4) is independent of the order of coupling the various angular momenta $\vec{R}_a$.

The richness of properties of the Jacobi coordinates (2.2) is not yet enough to facilitate the discussion of the permutational symmetry of the FPHO states. For this purpose it is convenient to introduce the Kramer-Moshinsky coordinates which are defined by

$$\vec{y}_1 = 1/2\ (\vec{x}_1+\vec{x}_4-\vec{x}_2-\vec{x}_3) \quad , \tag{2.5a}$$

$$\vec{y}_2 = 1/2\ (\vec{x}_2+\vec{x}_4-\vec{x}_1-\vec{x}_3) \quad , \tag{2.5b}$$

$$\vec{y}_3 = 1/2\ (\vec{x}_3+\vec{x}_4-\vec{x}_1-\vec{x}_2) \quad , \tag{2.5c}$$

$$\vec{y}_4 = \vec{x}_d \tag{2.5d}$$

We shall denote the FPHO states, with L=0, in the Kramer-Moshinsky coordinates by the ket

$$|n_1\ell_1,n_2\ell_2,n_3\ell_3) \quad . \tag{2.6}$$

This state has $n_4 = \ell_4 = 0$, allowing again no excitation of the center of mass, and is represented by a round ket) to distinguish between the angular ket > in (2.4).

It must be emphasised that the quantum numbers $n_i \ell_i$ in (2.6) affect the coordinates $\vec{y}$-in (2.5) and not the laboratory coordinates $\vec{x}_i$.

The relative Kramer-Moshinsky coordinates and the relative Jacobi ones are related between themselves through an orthogonal transformation, so the total number of quanta (2.3) is also given by

$$N = 2n_1+\ell_1+2n_2+\ell_2+2n_3+\ell_3 \ . \tag{2.7}$$

Explicitly, we have the following relation

$$\begin{pmatrix} \vec{y}_1 \\ \vec{y}_2 \\ \vec{y}_3 \end{pmatrix} = \begin{pmatrix} \sqrt{1/2} & \sqrt{1/6} & -\sqrt{1/3} \\ -\sqrt{1/2} & \sqrt{1/6} & -\sqrt{1/3} \\ 0 & -\sqrt{2/3} & -\sqrt{1/3} \end{pmatrix} \begin{pmatrix} \vec{x}_a \\ \vec{x}_b \\ \vec{x}_c \end{pmatrix} \ . \tag{2.8}$$

The parity of state (2.6) is given by the parity of N in (2.7). As the parity of the ground state of the ^{16}O, is even, we must have $\ell_1+\ell_2+\ell_3$ even. So, the three ℓ's must be even or two of them can be odd. This last possibility is to be disregarded if we want (2.6) to be completely symmetric[5]. Thus, all the angular momenta in (2.6) are even.

Finally, the state ϕ_ν with the properties mentioned in the beginning of this section is given by .

$$\begin{aligned} \phi_\nu = A_\nu [& |n_1\ell_1,n_2\ell_2,n_3\ell_3) + |n_1\ell_1,n_3\ell_3,n_2\ell_2) \\ & + |n_2\ell_2,n_1\ell_1,n_3\ell_3) + |n_2\ell_2,n_3\ell_3,n_1\ell_1) \\ & + |n_3\ell_3,n_1\ell_1,n_2\ell_2) + |n_3\ell_3,n_2\ell_2,n_1\ell_1) \\ & \equiv |n_1\ell_1,n_2\ell_2,n_3\ell_3)_S \ , \end{aligned} \tag{2.9}$$

where the normalization coefficient A_ν is such that

$$A_\nu = \begin{cases} \sqrt{1/6} & \text{if all pairs } (n_i \ell_i) \text{ are different} \\ 1/6 & \text{if all pairs } (n_i \ell_i) \text{ are equal} \\ \sqrt{1/12} & \text{otherwise .} \end{cases} \qquad (2.10)$$

3. MATRIX ELEMEMTS

In the last section, we have discussed the FPHO states in terms of which we expanded our variational function. It was mentioned that the Kramer-Moshinsky coordinates *(2.5)* are convenient to construct FPHO states with defined permutational symmetry and parity. Nevertheless, the Jacobi coordinates *(2.2)* are suitable for the calculation of the matrix elements of two-body and three-body operators as well as for the calculation of the form factors. So a linear transformation from the states *(2.6)* to the states *(2.4)* is rather convenient. The coefficients associated with this transformation are those in the following expression $(i,j,k = 1,2,3)$

$$|n_i \ell_i, n_j \ell_j, n_k \ell_k) =$$

$$\Sigma\, |n_a \ell_a, n_b \ell_b, n_c \ell_c > \langle n_a \ell_a, n_b \ell_b, n_c \ell_c |n_i \ell_i, n_j \ell_j, n_k \ell_k) \qquad (3.1)$$

where the sum is extended over all non negative integer values of n_a, ℓ_a, n_b, ℓ_b, n_c and ℓ_c subjected to the condition (2.7) and to the condition

$$\vec{\ell}_a + \vec{\ell}_b + \vec{\ell}_c = 0 \qquad (3.2)$$

The bracket $|n_a \ell_a, n_b \ell_b, n_c \ell_c \overset{R}{|} n_i \ell_i, n_j \ell_j, n_k \ell_k)$ is known in general[5,6]. In the case of $L=0$ it reduces to

$$\langle n_a \ell_a, n_b \ell_b, n_c \ell_c |n_i \ell_i, n_j \ell_j, n_k \ell_k) =$$

$$(-)^{\ell_c} \sum_{n\ell} \langle n_a \ell_a, n\ell, \ell_k |n_i \ell_i, n_j \ell_j, \ell_k) \langle n_b \ell_b, n_c \ell_c, \ell_a |n\ell, n_k \ell_k, \ell_a)_\beta \ . \qquad (3.3)$$

The brackets appearing in the RHS of (3.3) are Moshinsky coefficients[5]. The generalized Moshinsky coefficients are associated with the angle β defined by

$$\cos(\beta/2) = \sqrt{1/3} \ , \quad \sin(\beta/2) = \sqrt{2/3} \ . \tag{3.4}$$

The properties of these ccefficients are discussed in Ref.7.

In the following, we shall discuss the hamiltonian whose matrix elements between states (3.1) we are interested in.

Let us consider the four-particle hamiltonian

$$H = (\hbar\omega/2) \sum_{s=1}^{4} p_s^2 + \sum_{s<t=2}^{4} \left[V_{\alpha\alpha}(s,t) + V_c(s,t) \right]$$

$$+ \sum_{s<t<u} V_{3\alpha}(s,t,u) \tag{3.5}$$

The first term is the kinetic energy, V_c the Coulomb potential, $V_{\alpha\alpha}$ tha α-α interaction, and $V_{3\alpha}$ the 3α-potential.

The α-α potential we shall use in this paperwasconstructed hy Ali and Bodmer[8]. It is an ℓ-dependence phenomenological potential whose parameters were fitted as to reproduce the phase shifts δ_0, δ_2 and δ_4 associated with a-a scattering. Explicitly it is given by a superposition of attractive and repulsive gaussians as

$$V_{\alpha\alpha}(s,t) = \sum_{\ell=0,2,4} \left[V_{R\ell} \exp(-\mu_{R\ell}^2 r_{st}^2) + V_{A\ell} \exp(-\mu_{A\ell}^2 r_{st}^2) \right] . \tag{3.6}$$

Ali and Bodmer adjusted several sets of parameters. The ona we shall use here is shown in Table 1.

The 3α-potential is described by an attractive gaussian

$$V_{3\alpha}(s,t,u) = V_0 \exp\left[-\lambda(r_{st}^2 + r_{su}^2 + r_{tu}^2)\right] . \tag{3.7}$$

For this component of the potential energy, we considered two recently constructed potentials[1,2] whose parameters were adjusted in distinct

ℓ	$V_{R\ell}$ (MeV)	$\mu_{R\ell}$	$V_{A\ell}$ (MeV)	$\mu_{A\ell}$
0	475	0.7	- 130	0.475
2	320	0.7	- 130	0.475
4	10	0.7	- 130	0.475

Table 1 - Parameters of the Ali-Bodmer potential considered in this paper.

forms. Both potentials were adjusted in order to reproduce the ground-state energy of the ^{12}C nucleus. Additionally, Portilho and Coon required it to describe the energy gap of the lower 0^+ and 2^+ states, while Ogasawara and Hiura required the description of the radius of that nucleus. These different criteria led to potentials of very distinct characteristics as it is shown in table 2.

As the basic states are completely symmetric, for effect of matrix elements evaluation we may take for the potential energy the expression

$$V_{\alpha\alpha}+V_c+V_{3\alpha} = 6\left[V_{\alpha\alpha}(1,2) + V_c(1,2)\right] + 4V_{3\alpha}(1,2,3) , \qquad (3.8)$$

where the factors 6 and 4 account for the number of pairs and trios of particles, respectively.

We are interested in the intrinsic hamiltonian H_I which is

	V_0 (MeV)	λ (fm^{-2})
Portilho and Coon	- 7	5.06×10^{-3}
Ogasawara and Hiura	- 200	0.25

Table 2 - Parameters of the 3-a potentials.

obtained from (3.5) by subtracting the center-of-rnass kinetic energy, namely

$$H_I = H - (\hbar\omega/8) \left| \sum_{s=1}^{4} \vec{p}_s \right|^2 . \qquad (3.9)$$

Now, it is convenient to add and subtract from (3.9) the term

$$(\hbar\omega/8) \sum_{s,t=1}^{4} (\vec{x}_s - \vec{x}_t)^2 = \hbar\omega(x_a^2 + x_b^2 + x_c^2) , \qquad (3.10)$$

so that (3.9) can be expressed entirely in terms of Jacobi coordinates as

$$\begin{aligned} H_I = {} & \frac{1}{2}\hbar\omega \left| p_a^2 + p_b^2 + p_c^2 + x_a^2 + x_b^2 + x_c^2 \right| \\ & + 6 \left| V_{\alpha\alpha}(x_a) + V_c(x_a) - (\hbar\omega/4)x_a^2 \right| \\ & + 4\, V_{3\alpha}(x_a, x_b) . \end{aligned} \qquad (3.11)$$

The first term represents the hamiltonian of 3 harmonic oscillators with eigenvalue

$$(N + 9/2)\hbar\omega , \qquad (3.12)$$

where N is given by (2.7).

If we represent by N the set of quantum numbers of the basic states (2.6), i.e.,

$$N \equiv (n_i \ell_i, n_j \ell_j, n_k \ell_k) , \quad i,j,k = 1, 2, 3 \qquad (3.13)$$

we have to evaluate the matrix elements

$$\begin{aligned} (N'|H_I|N) = {} & \hbar\omega(N + 9/2)\, \delta_{N'N} + \\ & 6(N'|V_{\alpha\alpha}(x_a) + V_c(x_a) - (\hbar\omega/4)x_a^2|N) + \\ & 4(N'|V_{3\alpha}(x_a, x_b)|N) . \end{aligned} \qquad (3.14)$$

Essentially, we have to evaluate two types of matrix elements, namely

$$(n_i'\ell_i', n_j'\ell_j', n_k'\ell_k' | V_{\alpha a}(x_a) + V_c(x_a) - (\hbar\omega/4)\, x_a^2 | n_i\ell_i, n_j\ell_j, n_k\ell_k) \quad , \tag{3.15}$$

and

$$(n_i'\ell_i', n_j'\ell_j', n_k'\ell_k' | V_{3\alpha}(x_a, x_b) | n_i\ell_i, n_j\ell_j, n_k\ell_k) \, . \tag{3.16}$$

To evaluate these matrix elements it is obviously convenient to express bras and kets in Jacobi coordinates. This can be accomplished by the use of the transformation (3.1) and it is easy to see that the two-body contributions *are given* by

$$\begin{aligned}
&(n_i'\ell_i', n_j'\ell_j', n_k'\ell_k' \mid V_{\alpha a} + V_c - (\hbar\omega/4)x_a^2 | n_i\ell_i, n_j\ell_j, n_k\ell_k) = \\
&= \delta_{n_k' n_k}\, \delta_{\ell_k' \ell_k} \sum_{\substack{n_a'\ell_a' n_a \\ n\ell}} \langle n_a'\ell_a' \| V_{\alpha a} + V_c - (\hbar\omega/4)x_a^2 \| n_a\ell_a \rangle \\
&\cdot \langle n_a'\ell_a', n\ell, \ell_k | n_i'\ell_i', n_j'\ell_j', \ell_k \rangle \langle n_a\ell_a, n\ell, \ell_k | n_i\ell_i, n_j\ell_j, \ell_k \rangle \quad ,
\end{aligned} \tag{3.17}$$

where the summations are restrict by the properties of the Moshinsky coefficients. The "reduced" matrix elements can be expressed in terms of Talmi integrals[5] resulting

$$\begin{aligned}
&\langle n_a'\ell_a' \| V_{\alpha a} + V_c - (\hbar\omega/4)\, x_a^2 \| n_a\ell_a \rangle = \\
&\sum_{p=\ell_a}^{n_a'+n_a+\ell_a} B(p, n_a'\ell_a', n_a\ell_a) \left\{ \sum_{\ell=0,2,4} \left[V_{R\ell}(1+\alpha_{R\ell}/\varepsilon)^{-p-3/2} + \right.\right. \\
&\left.\left. V_{A\ell}(1+\alpha_{A\ell}/\varepsilon)^{-p-3/2} \right] + \frac{\kappa\sqrt{\varepsilon}\, p!}{\Gamma(p+3/2)} - \frac{\varepsilon}{4}(p+3/2) \right\}
\end{aligned} \tag{3.18}$$

with

$$\begin{aligned}
\varepsilon &= \hbar\omega/m_0c^2 \quad , & \kappa &= (e^2/\hbar c)(8m_\alpha/m_0)^{1/2} \\
\alpha_{R\ell, A\ell} &= 2\beta\mu^2_{R\ell, A\ell} \quad , & \beta &= (\hbar c)^2/m_\alpha c^2 m_0 c^2 \, .
\end{aligned} \tag{3.19}$$

In (3.19), m_0c^2 = 0.511 MeV is an arbitrary scale factor.

We note that the matrix elements (3.17) are invariant under the exchange of the pairs $(n_i\ell_i) \leftrightarrow (n_3\ell_3)$ and/or $(n'_i\ell'_i) \leftrightarrow (n'_3\ell'_3)$ due to symmetry properties of the Moshinsky coefficients. This fact reduces drastically the number of matrix elements to be effectively evaluated.

The matrix elements (3.16) are written as

$$(n'_i\ell'_i, n'_j\ell'_j, n'_k\ell'_k \,|V_{3\alpha}(x_a,x_b)|\, n_i\ell_i, n_j\ell_j, n_k\ell_k) =$$

$$\sum_{\substack{n_a\ell_a \\ n'_a\ell'_a}} \sum_{\substack{n_b\ell_b \\ n'_b\ell'_b}} \sum_{\substack{n_d\ell_d \\ n'_d\ell'_d}} \sum_{n_c\ell_c} \langle n_a\ell_a, n_d\ell_d, \ell_k | n_i\ell_i, n_j\ell_j, \ell_k\rangle$$

$$\times \langle n'_a\ell'_a, n'_d\ell'_d, \ell'_k | n'_i\ell'_i, n'_j\ell'_j, \ell'_k\rangle \langle n_b\ell_b, n_c\ell_c, \ell_a | n_d\ell_d, n_k\ell_k, \ell_a\rangle_\beta$$

$$\times \langle n'_b\ell'_b, n_c\ell_c, \ell'_a | n'_d\ell'_d, n'_k\ell'_k, \ell'_a\rangle_\beta \langle n'_a\ell'_a, n'_b\ell'_b \| V_{3\alpha} \| n_a\ell_a, n_b\ell_b\rangle , \qquad (3.20)$$

where the summations are restricted by properties of the several Moshinsky coefficients involved.

Since we can factorize the 3-body operator as

$$V_{3\alpha}(x_a,x_b) = V_0\, R^{-3\lambda(x_a^2+x_b^2)} \equiv V_0\, f(x_a) f(x_b) , \qquad (3.21)$$

the corresponding reduced matrix elements can be written as

$$\langle n'_a\ell'_a, n'_b\ell'_b \| V_{3\alpha} \| n_a\ell_a, n_b\ell_b\rangle =$$

$$V_0 \langle n'_a\ell'_a \| e^{-3\lambda x_a^2} \| n_a\ell_a\rangle \langle n'_b\ell'_b \| e^{-3\lambda x_b^2} \| n_b\ell_b\rangle \delta_{\ell'_a\ell_a} \delta_{\ell'_b\ell_b}$$

$$= V_0 \delta_{\ell'_a\ell_a} \delta_{\ell'_b\ell_b} \sum_{pq} B(p, n'_a\ell_a, n_a\ell_a) B(q, n'_b\ell_b, n_b\ell_b)$$

$$\times (1 + 3\lambda\beta/\epsilon)^{-p-q-3} \qquad (3.22)$$

with

$$\ell_a \leq p \leq n'_a + n_a + R_a \text{ and } \ell_b \leq q \leq n'_b + n_b + \ell_b \ . \quad (3.23)$$

Again, we note that the matrix elements (3.20) are invariant under the exchanges $(n_i\ell_i) \leftrightarrow (n_j\ell_j)$ and/or $(n'_i\ell'_i) \leftrightarrow (n'_j\ell'_j)$. In spite of the economy this symmetry provides, the presence of generalized Moshinsky coefficients requires too much cornputer time. In order to save computer time, the following development was carried out.

Let us introduce the following orthogonal transformation

$$\begin{aligned} \vec{y}_1 &= \vec{y}_1 \\ \vec{y} &= \sqrt{1/2}\ (\vec{y}_2 - \vec{y}_3) \\ \vec{Y} &= \sqrt{1/2}\ (\vec{y}_2 + \vec{y}_3) \end{aligned} \quad (3.24)$$

where $\vec{y}_i$ are the Kramer-Moshinsky coordinates (2.5). It is easy to see that

$$x_a^2 + x_b^2 = y^2 + \frac{2}{3} y_1^2 + \frac{1}{3} Y^2 - \frac{\sqrt{8}}{3} \vec{y}_i \ . \ \vec{Y} \quad (3.25)$$

so that (3.21) reads[g]

$$\begin{aligned} V_{3\alpha}(x_a, x_b) &= V_0\ e^{-3\lambda y^2} e^{-\lambda(\sqrt{2}\,\vec{y}_1 - \vec{Y})^2} \\ &= 4\pi\ V_0\ e^{-3\lambda y^2} \quad e^{-\lambda(2y_1^2 + Y^2)} \\ &\times \sum_{\ell m} i^{-\ell} j_\ell(i\sqrt{8}\lambda y_1 Y) Y_{\ell m}(\Omega_{\vec{y}_1}) Y^*_{\ell m}(\Omega_{\vec{y}_2}) \ . \end{aligned} \quad (3.26)$$

The oscillators in the coordinates $\vec{y}_2$ and $\vec{y}_3$ are connected with those in the coordinates $\vec{y}$ and $\vec{Y}$ through ordinary Moshinsky transformation brackets, so that we can write

$$|n_i\ell_i, n_j\ell_j, n_k\ell_k) = \sum_{\substack{n\ell \\ NL}} |n_i\ell_i, n\ell, NL)(n\ell, NL, \ell_i | n_j\ell_j, n_k\ell_k, \ell_i) \ , \quad (3.27)$$

where $(n\ell)$ and (NL) are associated with the coordinates $\vec{y}$ and $\vec{Y}$, respectively.

Now a little Racah algebra leads us to the final expression

$$(n'_t\ell'_t, n'_j\ell'_j, n'_k\ell'_k | V_{3a}(x_a, x_b) | n_i a_i, n_j a_j, n_k \ell_k) =$$

$$\sum_{\substack{n\ell \\ NL}} \sum_{\substack{n'\ell' \\ N'L'}} \sum_{pq} (\ell_i 0 \ell' 0 | \ell'_i 0)(L 0 \ell' 0 | L' 0) 2^{\ell'/2} (n'\ell' \| e^{-3\lambda y^2} \| n\ell)$$

$$\times \left[(2\ell_i + 1)(2L + 1)\right]^{1/2} (2\ell' + 1) \cdot (n\ell, NL, \ell_i | n_j \ell_j, n_k \ell_k, \ell'_i)$$

$$\times (n'\ell', N'L', \ell'_i | n'_j \ell'_j, n'_k \ell'_k, \ell'_i) \cdot B(p, N'L', NL) B(q, n'_i \ell'_i, n_i \ell_i)$$

$$\times W(\ell'_i L' \ell_i L; \ell\ell')$$

$$\times \frac{\lambda^{\ell'} \Gamma(3/2)\Gamma(p+\ell'/2+3/2)\Gamma(q+\ell'/2+3/2)}{\Gamma(q+3/2)\Gamma(p+3/2)\Gamma(\ell'+3/2)(\lambda+1)^{p+\ell'/2+3/2}(2\lambda+1)^{q+\ell'/2+3/2}}$$

$$\times {}_2F_1\left[p+\ell'/2+3/2,\ q+R'/2+3/2;\ R'+3/2;\ 2\lambda^2/(\lambda+1)(2\lambda+1)\right] . \qquad (3.28)$$

Taking into account the properties of the Clebsch-Gordan and Moshinsky coefficients it can be shown that all the i's involved in the summations are even. Finally, we note that (3.28) is invariant underthe exchanges $(n_3 R_3) \leftrightarrow (n_k \ell_k)$ and/or $(n'_3 \ell'_3) \leftrightarrow (n'_k \ell'_k)$.

The above expression (3.28) is much more complicated to write down than theequivalent one given in (3.22). Nevertheless, its use allowed us to reduce computer time to less than half the previous time!

The discussion of the matrix elements of the hamiltonian is complete. In the next section, we discuss the charge form factor.

4. THE CHARGE FORM FACTOR OF ^{16}O

It is well kncwn that the charge form factor of the ^{16}O nucleus factorizes as the product of its body forrn factor by the alpha particle charge form factor, i.e,

$$F_{ch}(\vec{q}) = F_b(\vec{q})\, f_\alpha(\vec{q}) \tag{4.1}$$

As we are here interested only in the effect of three-alpha forces in the charge form factor and since the alpha-particle charge form factor is obviously independent of the interactions between the alphas, we shall restrict ourselves to the discussion of the ^{16}O body form factor. If the ground state is described by the variational function (2.1), then the body form factor is given by[5]

$$F_b(\vec{q}) = \sum_{\nu'\nu} a^*_{\nu'}\, a_\nu \langle \nu' | j_0(kx_c) | \nu \rangle \tag{4.2}$$

whete v stands for the set $(n_1\ell_1, n_2\ell_2, n_3\ell_3)$,

$$\vec{k} = (3\beta/4\varepsilon)^{1/2}\, \vec{q} \tag{4.3}$$

and 8, ε are defined in (3.19).

On the coefficients a were deterrnined through the variational analysis of the hamiltonian, it rernains only to evaluate the matrix elements that appear in (4.2). A specirnen of them is given by

$$\begin{aligned}
&\langle n'_i\ell'_i, n'_j\ell'_j, n'_k\ell'_k | j_0(kx_c) | n_i\ell_i, n_j\ell_j, n_k\ell_k \rangle = \\
&\quad \sum (-)^{\ell_c+\ell'_c} \langle n'_a\ell'_a, n'_d\ell'_d, \ell'_k | n'_i\ell'_i, n'_j\ell'_j, \ell'_k \rangle \\
&\times \langle n_a\ell_a, n_d\ell_d, \ell_k | n_i\ell_i, n_j\ell_j, \ell_k \rangle \langle n'_b\ell'_b, n'_c\ell'_c, \ell_a | n'_d\ell'_d, n'_b\ell'_b, \ell_a \rangle_\beta \\
&\times \langle n_b\ell_b, n_c\ell_c, \ell_a | n_d\ell_d, n_k\ell_k, \ell_a \rangle_\beta \langle n'_c\ell'_c \| j_0(kx_c) \| n_c\ell_c \rangle \\
&\times \delta_{n'_a n_a} \delta_{\ell'_a \ell_a} \delta_{n'_b n_b} \delta_{\ell'_b \ell_b} \delta_{\ell'_c \ell_c}
\end{aligned} \tag{4.4}$$

where use was made of the transformation (3.1). In (4.4) the summation runs over all the quantum number that do not appear in the LHS. There ai-e 16 summations. The presence of the several Kronecker symbols and the properties of the Moshinsky coefficients leave us with only 7 independent summations.

The expression (4.4), so easy to be written down, also consumes ton much computer time. Thus, we had to find an alternative expression which does not involve generalized Moshinsky brackets. The procedure is somewhat similar to that developed for the evaluation of the matrix elements of the 3-body operators in last section.

From *(2.8)* we see that

$$\vec{x}_c = -\sqrt{1/3}\ (\vec{y}_1+\vec{y}_2+\vec{y}_3) \tag{4.5}$$

Now, using the expansion

$$j_0(\alpha|\vec{r}-\vec{\rho}|) = 4\pi \sum_{\lambda\mu} j_\lambda(\alpha r) j_\lambda(\alpha\rho) Y^*_{\lambda\mu}(\Omega_{\vec{r}}) Y_{\lambda\mu}(\Omega_{\vec{\rho}}) \tag{4.6}$$

we obtain

$$\begin{aligned} j_0(kx_c) &= j_0(\alpha y_1) j_0(\alpha y_2) j_0(\alpha y_3) \\ &+ 4\pi\, j_0(\alpha y_1) \sum_{\lambda\mu}{}' (-)^\lambda\, j_\lambda(\alpha y_2) j_\lambda(\alpha y_3) Y^*_{\lambda\mu}(\Omega_{\vec{y}_2}) Y_{\lambda\mu}(\Omega_{\vec{y}_3}) \\ &+ 4\pi \sum_{\lambda\mu}{}' (-)^\lambda\, j_\lambda(\alpha y_1) j_\lambda(\alpha\rho) Y^*_{\lambda\mu}(\Omega_{\vec{y}_1}) Y_{\lambda\mu}(\Omega_{\vec{\rho}}) \\ &\equiv S_1 + S_2 + S_3, \end{aligned} \tag{4.7}$$

with $\vec{\rho} = \vec{y}_2+\vec{y}_3$ and $a = -k/\sqrt{3}$. The prime in the summation symbol is to mean that the term X=0 is excluded from the sum.

Once the indices 1,2 and 3 are separated in S_1 and S_2 the corresponding contributions can be evaluated easily after some Racah algebra giving

$$E_1 \equiv (n_i'\ell_i', n_j'\ell_j', n_k'\ell_k' | S_1 | n_i\ell_i, n_j\ell_j, n_k\ell_k)$$

$$= \delta_{\ell_i'\ell_i} \delta_{\ell_j'\ell_j} \delta_{\ell_k'\ell_k} (n_i'\ell_i' \| j_0(\alpha y_1) \| n_i\ell_i)$$

$$\times (n_j'\ell_j' \| j_0(\alpha y_2) \| n_j\ell_j) (n_k'\ell_k' \| j_0(\alpha y_3) \| n_k\ell_k) \tag{4.8}$$

and

$$E_2 \equiv (n_i'\ell_i', n_j'\ell_j', n_k'\ell_k' | S_2 | n_i\ell_i, n_j\ell_j, n_k\ell_k)$$

$$= \delta_{\ell_i'\ell_i} \left[(2\ell_j+1)(2\ell_k+1)\right]^{1/2} (n_i'\ell_i' \| j_0(\alpha y_1) \| n_i\ell_i)$$

$$\times \sum_{\substack{\lambda \neq 0 \\ \text{(even)}}} (2\lambda+1)(\lambda 0 \ell_j 0 | \ell_j' 0)(\lambda 0 \ell_k 0 | \ell_k' 0)$$

$$\times W(\ell_j \ell_k \ell_j' \ell_k'; \ell_i \lambda) (n_j'\ell_j' \| j_\lambda(\alpha y_2) \| n_j\ell_j) (n_k'\ell_k' \| j_\lambda(\alpha y_3) \| n_k\ell_k)$$

(4.9)

To evaluate the matrix elements of S_3 we use again the transformation (3.24) and some Racah algebra to get

$$E_3 \equiv (n_i'\ell_i', n_j'\ell_j', n_k'\ell_k' | S_3 | n_i \mathbf{R}_i, n_j \mathbf{R}_k \mathbf{R}_k R)$$

$$= \sum_{\substack{\lambda \neq 0 \\ \text{(even)}}} (2\lambda+1)(n_i'\ell_i' \| j_\lambda(\alpha y_1) \| n_i\ell_i) \sum_{\substack{n\ell \\ NL \\ N'L'}} \langle n\ell, NL, \ell_i | n_j\ell_j, n_k\ell_k, \ell_i \rangle$$

$$\times \langle n\ell, N'L', \ell_i' | n_j'\ell_j', n_k'\ell_k', \ell_i \rangle (-)^{L'-\ell} (\ell_i 0 \lambda 0 | \ell_i' 0)(L 0 \lambda 0 | L' 0)$$

$$\times \left|(2L+1)(2\ell_i+1)\right|^{1/2} W(\ell_i'\ell_i L'L; \lambda\ell) (N'L' \| j_\lambda(\alpha\sqrt{2}y) \| NL)$$

(4.10)

Because all ℓ_i, L_3 and ℓ_k are even, and the Gaunt coefficients $(a0b0|c0)$ require that $a+b+c$ = even, the A in (4.9) and (4.10) is even.

An alternative equivalente expression for (4.4) is then gi-

ven by $E_1+E_2+E_3$. All it remains to evaluate are the reduced matrix elements of j_0. They are easily obtained through the use of Talmi integral and are given by

$$(n'\ell'\| j_0(\beta x)\| n\ell)$$

$$= \sum_p B(p,n'\ell',n\ell)(\beta/2)^\lambda \frac{\Gamma(3/2)\Gamma(p+\lambda/2+3/2)}{\Gamma(\lambda+3/2)\Gamma(p+3/2)} M(p+\frac{\lambda}{2}+\frac{3}{2},\lambda+\frac{3}{2},-\frac{\beta^2}{4}), \tag{4.11}$$

where $M(a,b,z)$ is a Kummer function and p runs over the values

$$\frac{1}{2}(\ell'+\ell) \leq p \leq \frac{1}{2}(\ell'+\ell) + n' + n . \tag{4.12}$$

Again, we obtained an expression much more complicated when we tried to avoid the generalized Moshinsky coefficients. However, expressions (4.8) - (4.10) help us to save a lot of computer time.

5. RESULTS AND DISCUSSION

Using only the two-alpha interaction (3.6) with the parameters of Table I, we found a ground-state energy of -4.8 MeV for the ^{16}O nucleus. In this case, the numerical calculations wsre carried out in the 10-quantum approximation what means that the variational function (2.1) has 40 components. The above energy agrees with the one previously obtained by Mendez-Moreno et $al.^4$. The experimental value is -14.5 MeV. So, the u-a potential alone provides cnly 30% of the experimental value in that approximation.

Turning on the 3-a potential (3.7), we learn that the Portilho-Coon potential with the parameters of Table 2 overbinds the system early in the 6-quantum approximation, surely on account of its long range character. On the other hand, the potential of Ogasawara-Hiura, whose parameters are listed in Table 2, gives -4.74 MeV in the 8-quantum approximation. It must be stressed that with the help of this 3-α force we required a subspace much smaller (half the dimension) than that required by using only 2-u potential to obtain the sanie energy.

This is significant and gives us a clear demonstration of the effectiveness of the 3-α force in the ^{16}O nucleus.

Numerical calculations up to a greater number of quanta in the approximation - that would allow a reliable discussion on excited states and inelastic form factor, for instance - were not carried out because of the very expensive coaputer costs in spite of the reduction in process time as mentioned before.

Concerning the ^{16}O form factor, only the body form factor is presented because we presently are interested only in verify the effect of a 3-α force. This is shown in Fig.1. The dashed line is obtained without the presence of a three-alpha force. When we turn this force on we obtain the body form factor represented by the continuous line. We see from the figure that the form factor is very insensitive to the presence of the 3-α force for small momentum transfer $\hbar q$. A slight effect can be noted according as $\hbar q$ increases.

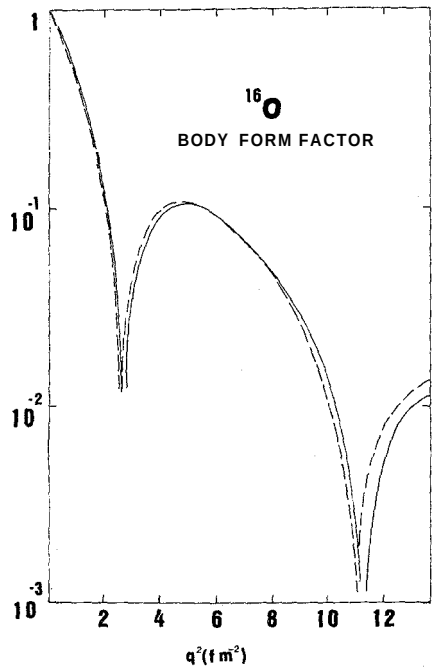

Fig.1 - Body form factor of ^{16}O with Ali-Bodmer + Ogasawara-Hiura potentials. Dashed line: 3-α force turned off. Continuous line: 3-α force turned on.

The authors thanks very much Prof. O. Portilho for his help at the Computational Center of the Universidade de Brasília.

REFERENCES

1. H.Ogasawara and J.Hiura, Prog.Theor.Phys. 59, 655 (1978).

2. O.Portilho and S.A.Coon, Z.Physik A*290*, 93 (1979).

3. V.C.Aguilera-Navarro, O.Portilho and R.Yamaoka, Rev.Bras.Fis.*4*,107 (1974).

4. R.M.Méndez-Moreno, M.Moreno and T.H.Seligman, Nucl.Phys. A*221*, 381 (1974).

5. M.Moshinsky, *The Harmonic Oscillator in Modern Physics: From Atoms to Quarks*, Gordon & Breach, New York, 1969.

6. V.C.Aguilera-Navarro, Rev.Mex.Fis. 20, 31 (1971).

7. V.C.Aguilera-Navarro, T.A.Brody and J.Flores, Rev.Mex.Fis.*19*, 303 (1970).

8. S.Ali and A.R.Bodmer, Nucl.Phys. 80, 99 (1966).

9. D.M. Brink and G.R.Satchler, *Angular Momentum*, Clarendon Press, Oxford, 1968, p.151.

10. M.Abramowitz and I.A.Stegun, *Handbook of Mathematical Functions*, Dover Publ., N.Y. p.440 (1965).